

PARTIAL DIFFERENTIAL OPERATORS WITH NON-NEGATIVE CHARACTERISTIC FORM, MAXIMUM PRINCIPLES, AND UNIQUENESS FOR BOUNDARY VALUE AND OBSTACLE PROBLEMS

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ABSTRACT. We prove weak and strong maximum principles, including a Hopf lemma, for classical solutions to equations defined by linear, second-order, partial differential operators with non-negative characteristic form (degenerate-elliptic operators), in the presence of a second-order boundary condition of Ventcel type along the degeneracy locus of the principle symbol of the operator on the domain boundary. We apply these maximum principles to obtain uniqueness and a priori maximum principle estimates for classical solutions to boundary value and obstacle problems defined by these degenerate-elliptic operators, again in the presence of a second-order boundary condition, for Dirichlet or Neumann boundary conditions along the complement of the degeneracy locus. We also prove weak maximum principles and uniqueness for solutions to the corresponding variational equations and inequalities defined with the aid of weighted Sobolev spaces. The domain is allowed to be unbounded when the operator coefficients and solutions obey certain growth conditions.

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1. INTRODUCTION

The classical maximum principles of Fichera [34, 35] and Oleřnik, and Radkevič [58, 63, 64] provide uniqueness theorems for degenerate elliptic and parabolic boundary value problems which do not take into account a more modern view of the appropriate function spaces in which uniqueness is sought, such as [18, 19, 20, 26, 27, 31, 30, 47]. Indeed, their maximum principles lead to the imposition of additional Dirichlet boundary conditions which are not necessarily motivated by the underlying application, whether in biology, finance, or physics. These additional Dirichlet boundary conditions, usually for certain ranges of parameters defining the partial differential

equation, are often less natural than the physically-motivated regularity properties suggested by choices of appropriate weighted Hölder [19, 20, 27, 30] or Sobolev spaces [18, 31, 47], which automatically encode special regularity or integrability up to portions of the domain boundary where the operator becomes degenerate. In the case of weighted Hölder spaces, these boundary regularity properties may be viewed as special cases of *second-order* or *Ventcel boundary conditions* [4, 5, 68](see Remark 2.5). However, the question of exactly how regular the solution should be near these boundary portions is delicate: asking for too much regularity (such as C^2 up to the boundary) and the boundary value problem may not be well-posed, while asking for too little regularity (such as C^0 up to the boundary) may lead to the requirement of an unphysical Dirichlet boundary to ensure the problem is well-posed, with the unintended consequence that the solutions thus selected can be no more than continuous up to the boundary; an illustration of this point is provided by Example 1.3.

1.1. Formulation of boundary value and obstacle problems for linear, second-order partial differential operators with non-negative characteristic form. Let $\mathcal{O} \subset \mathbb{R}^d$, $d \geq 2$, be a domain, that is, a connected, open, possibly unbounded subset with boundary $\partial\mathcal{O}$ and suppose that $\Sigma \subseteq \partial\mathcal{O}$ is a relatively open subset. We shall consider the question of uniqueness and a priori estimates for the boundary value problem,

$$\begin{aligned} Au &= f \quad \text{on } \mathcal{O}, \\ u &= g \quad \text{on } \partial\mathcal{O} \setminus \Sigma, \end{aligned} \tag{1.1}$$

and for the obstacle problem,

$$\begin{aligned} \min\{Au - f, u - \psi\} &= 0 \quad \text{a.e. on } \mathcal{O}, \\ u &= g \quad \text{on } \partial\mathcal{O} \setminus \Sigma, \end{aligned} \tag{1.2}$$

where $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$, and

$$Au := -\text{tr}(aD^2u) - \langle b, Du \rangle + cu, \quad u \in C^\infty(\mathcal{O}), \tag{1.3}$$

where $a : \bar{\mathcal{O}} \rightarrow \mathbb{R}^{d \times d}$ has the property that¹ $a(x)$ is a non-negative definite matrix for all $x \in \bar{\mathcal{O}}$,

$$\langle a(x)\eta, \eta \rangle \geq 0, \quad \forall x \in \bar{\mathcal{O}}, \quad \forall \eta \in \mathbb{R}^d.$$

and choose²

$$\Sigma = \text{int}(\{x \in \partial\mathcal{O} : a(x) = 0\}),$$

where $\text{int}(S)$ denotes the interior of a subset S of a topological space. Throughout this article we shall allow Σ to be non-empty. Finally, $b : \bar{\mathcal{O}} \rightarrow \mathbb{R}^d$ is a vector field and $c : \bar{\mathcal{O}} \rightarrow \mathbb{R}$ is a non-negative function. We use D^2u and Du to denote the Hessian matrix and gradient of u , respectively, so $\text{tr}(aD^2u) = a^{ij}u_{x_i x_j}$ and $\langle b, Du \rangle = b^i u_{x_i}$, where Einstein's summation convention is used throughout this article. Following [58], we call A in (1.3) a linear, second-order partial differential operator with *non-negative characteristic form*. Such operators are more commonly known as “degenerate elliptic” when Σ is non-empty (see Remark 4.4 for definitions of elliptic, strictly elliptic, and uniformly elliptic operators).

¹Though unnecessary for the maximum principle, an additional assumption of symmetry ensures that we may find a square root, $\sigma : \bar{\mathcal{O}} \rightarrow \mathbb{R}^{d \times d}$ such that $a = \frac{1}{2}\sigma\sigma^*$ and consider the martingale problem for (a, b) [67].

²For the sake of notational consistency in this article, we do not always distinguish between situations where $\Sigma \subseteq \partial\mathcal{O}$ may be any open subset rather than the zero locus of the matrix a on $\partial\mathcal{O}$ and instead rely on the context to make the distinction clear.

For the sake of clarity, we shall confine our attention to linear operators in this article, although many of the results can be seen to extend to *semilinear* operators,

$$S(u) = -\operatorname{tr}(aD^2u) - \langle b, Du \rangle + c(\cdot, u), \quad u \in C^2(\mathcal{O}),$$

or *quasilinear* operators [40, §10],

$$Q(u) = -\operatorname{tr}(aD^2u) - b(\cdot, u, Du), \quad u \in C^2(\mathcal{O}),$$

as well as linear and quasilinear *parabolic* operators with non-negative characteristic form. We shall discuss such situations in separate articles.

The conditions on the coefficients of A in (1.3) assumed in this article are mild enough that they allow for many examples of partial differential operators, A , with non-negative characteristic form and which are of interest in mathematical biology, finance, and physics. Before proceeding to a discussion of our main results, we shall first provide some specific examples of operators to which our results apply.

1.2. Examples. We begin by describing a family of examples which includes certain stochastic volatility models occurring in mathematical finance [45] and the linearization of the porous medium equation [19].

Example 1.1 (Affine coefficients and degeneracy on the boundary of a half-space). Suppose the coefficients of A in (1.3) are affine functions of $x \in \mathbb{R}^d$, with $a(x)$ positive definite for all $x \in \mathbb{H}$, where $\mathbb{H} = \mathbb{R}^{d-1} \times \mathbb{R}_+$ is a half-space and $\mathbb{R}_+ = (0, \infty)$, while $a(x) = 0$ if $x \in \partial\mathbb{H}$. Then

$$Au = -\operatorname{tr}(x_d a_1 D^2 u) - \langle b_0 + b_1 x, Du \rangle + (c_0 + \langle c_1, x \rangle)u, \quad u \in C^\infty(\mathbb{H}), \quad (1.4)$$

where $a_1, b_1 \in \mathbb{R}^{d \times d}$ and $b_0, c_1 \in \mathbb{R}^d$ and $c_0 \in \mathbb{R}$. Thus, A is an elliptic partial differential operator on $C^\infty(\mathbb{H})$ which becomes degenerate along the boundary $\Sigma = \partial\mathbb{H} = \{x_d = 0\}$ of the half-space $\mathbb{H} = \{x_d > 0\}$.

When a_1 is symmetric, the operator $-A$ is the generator of a degenerate diffusion process. Imposing the condition $\langle b, \vec{n} \rangle \geq 0$ along $\partial\mathbb{H}$, where \vec{n} is the *inward*-pointing unit normal vector field, ensures that the diffusion process remains in the half-space $\bar{\mathbb{H}} = \{x_d \geq 0\}$ if started in $\bar{\mathbb{H}}$. Since $\vec{n} = e_d$, this translates to the requirement that $b^d(x) \geq 0$ for all $x \in \partial\mathbb{H}$, and thus $b_0^d \geq 0$ and $b_1^{dj} = 0$ for $1 \leq j \leq d-1$ and $b_1^{dd} \geq 0$.

To ensure uniqueness of solutions to (1.1) on \mathbb{H} via our maximum principle, it is necessary (though not sufficient) to impose the condition $c \geq 0$ on \mathbb{H} , and hence $c_0 \geq 0$ and $c_1^i = 0$ for $1 \leq i \leq d-1$ and $c_1^d \geq 0$.

Examples of this kind occur frequently in mathematical finance, such as the Heston stochastic volatility process [45], where $a(x) = x_d a_1$ for $x \in \mathbb{H}$, and $a_1 \in \mathbb{R}^{d \times d}$ is positive definite, and $d = 2$. See Example 1.2 for a description of the Heston process generator and an important example of a degenerate affine process. The linearization of the porous medium operator is another important example of this type; see Example 1.4 for details. \square

Example 1.2 (Elliptic Heston operator). The generator of the *Heston process* [45] provides a well-known example in mathematical finance of the operator (1.4) when $d = 2$:

$$Au := -\frac{x_2}{2} (u_{x_1 x_2} + 2\rho\sigma u_{x_1 x_2} + \sigma^2 u_{x_2 x_2}) - (r - q - x_2/2)u_{x_1} - \kappa(\theta - x_2)u_{x_2} + ru, \quad (1.5)$$

where $q \geq 0, r \geq 0, \kappa > 0, \theta > 0, \sigma > 0$, and $\rho \in (-1, 1)$ are constants (their financial interpretation is provided in [45]), and $u \in C^\infty(\mathbb{H})$, with $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$. \square

Example 1.3 (Elliptic Feller square-root or Cox-Ingersoll-Ross operator). The generator of the *Feller square-root* process [33], known as the *Cox-Ingersoll-Ross* process in mathematical finance [13], [14], [66, Example 6.5.2], provides a simple example of the operator (1.4) when $d = 1$:

$$Au := -\frac{\sigma^2}{2}xu_{xx} - \kappa(\theta - x)u_x + ru, \quad (1.6)$$

where $u \in C^\infty(\mathbb{R}_+)$, and which takes the form, after a change of variables, of the *Kummer* operator

$$Bv := -xv_{xx} - (\beta - x)v_x + \alpha v, \quad (1.7)$$

where $v \in C^\infty(\mathbb{R}_+)$ and $\beta := 2\kappa\theta/\sigma^2 > 0$ and $\alpha := r/\kappa \geq 0$. The homogeneous Kummer equation, $Bv = 0$ on \mathbb{R}_+ , has two independent solutions, the *confluent hypergeometric function of the first kind* $M(\alpha, \beta; x)$ (or Kummer function) and the *confluent hypergeometric function of the second kind* $U(\alpha, \beta; x)$ (or Tricomi function) [2, §13.1.2 & §13.1.3]. The solution $M(\alpha, \beta; x)$ is in $C_{\text{loc}}^\infty(\mathbb{R}_+)$, with $M(\alpha, \beta; 0) = 1$, $M_x(\alpha, \beta; 0) = \alpha/\beta$, and $M_{xx}(\alpha, \beta; 0) = \alpha(\alpha + 1)/(\beta(\beta + 1))$ [2, §13.4.9]. Near $x = 0$, the solution $U(\alpha, \beta; x)$ is comparable to $x^{1-\beta}$ when $\beta \neq 1$ and $\log x$ when $\beta = 1$ [2, §13.5.6–12], and thus is in $C_{\text{loc}}^0(\mathbb{R}_+)$ for $0 < \beta < 1$, with $U_x(\alpha, \beta; x)$ comparable to $x^{-\beta}$ and $U_{xx}(\alpha, \beta; x)$ comparable to $x^{-\beta-1}$ near $x = 0$, for any $\beta > 0$ [2, §13.4.22], and thus not even in $C_{\text{loc}}^1(\mathbb{R}_+)$ when $\beta > 0$.

Example 1.4 (Linearization of the porous medium operator). In a landmark article, Daskalopoulos and Hamilton [19] proved existence and uniqueness of C^∞ solutions, u , to the Cauchy problem for the porous medium equation [19, p. 899] (when $d = 2$),

$$-u_t + \sum_{i=1}^d (u^m)_{x_i x_i} = 0 \quad \text{on } (0, T) \times \mathbb{R}^d, \quad u(\cdot, 0) = g \quad \text{on } \mathbb{R}^d, \quad (1.8)$$

with constant $m > 1$ and initial data, $g \geq 0$, compactly supported on \mathbb{R}^d , together with C^∞ -regularity of its free boundary, $\partial\{u > 0\}$, provided the initial pressure function is non-degenerate (that is, $Du^{m-1} \geq a > 0$) on boundary of its support at $t = 0$. Their analysis is based on their development of existence, uniqueness, and regularity results for the linearization of the porous medium equation near the free boundary and, in particular, their *model linear degenerate operator* [19, p. 901] (generalized from $d = 2$ in their article),

$$Au := -x_d \sum_{i=1}^d u_{x_i x_i} - \beta u_{x_d}, \quad u \in C^\infty(\mathbb{H}), \quad (1.9)$$

where β is a positive constant, analogous to the combination of parameters $2\kappa\theta/\sigma^2$ in (1.5), and $\mathbb{H} = \mathbb{R}^{d-1} \times \mathbb{R}_+$, following a suitable change of coordinates [19, p. 941]. The same model linear degenerate operator (for $d \geq 2$), was studied independently by Koch [47, Equation (4.43)] and, in a remarkable Habilitation thesis, he obtained existence, uniqueness, and regularity results for solutions to (1.8) which complement those of Daskalopoulos and Hamilton [19]. \square

Example 1.1 describes a class of elliptic differential operators which become degenerate along the boundary of a half-space, $\mathbb{R}^{d-1} \times \mathbb{R}_+$; their coefficients are affine functions of $x \in \mathbb{R}^d$ and $\mathbb{R}^{d-1} \times \bar{\mathbb{R}}_+$ is a state space for the corresponding Markov process when the coefficient $a(x)$ is symmetric. More generally, mathematical finance and biology provide examples of elliptic differential operators which become degenerate along the boundary of a “quadrant”, $\mathbb{R}^{d-m} \times \mathbb{R}_+^m$, and $\mathbb{R}^{d-m} \times \bar{\mathbb{R}}_+^m$ is a state space for the corresponding Markov process. Examples primarily motivated by mathematical finance include affine processes [1, 12, 17, 22, 23, 24, 36, 37], which

may be viewed as extensions of geometric Brownian motion (see, for example, [66]), the Heston stochastic volatility process [45], and the Wishart process [12, 38, 41, 42, 43]. Examples of this kind which arise in mathematical biology include the multi-dimensional Kimura diffusions and their local model processes [27, Equations (1.5) & (1.20)]. Another example along these lines is provided by the articles of Athreya, Barlow, Bass, Lavrentiev, and Perkins [6, 8, 7] concerning generators of super-Markov chains.

Maximum principles addressing these more complex situations where A has a non-negative characteristic form which becomes degenerate along a *stratified space* $\bar{\Sigma} \subseteq \partial\mathcal{O}$, and in the broader sense that

$$\bar{\Sigma} = \{x \in \partial\mathcal{O} : \det a(x) = 0\},$$

rather than $\Sigma = \text{int}(\{x \in \partial\mathcal{O} : a(x) = 0\})$, are discussed in our companion article [29].

1.3. Summary of main results and outline of our article. We shall leave detailed statements of our main results to the body of our article and simply provide a short outline of our article here to facilitate the reader seeking a particular conclusion of interest. Part 1 of our article (§2, §3, §4, and §5) develops weak and strong maximum principles for operators on smooth functions and applications to boundary value and obstacle problems, while Part 2 of our article (§6 and §7) develops weak maximum principles for bilinear maps and operators on functions in Sobolev spaces and applications to variational equations and inequalities.

1.3.1. Weak and strong maximum principles for operators on smooth functions and applications to boundary value and obstacle problems. We say that $u \in C^2(\mathcal{O}) \cap C^1(\mathcal{O} \cup \Sigma) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeys a *second-order boundary condition along Σ* defined by a principal symbol $a \in C_{\text{loc}}(\mathcal{O} \cup \Sigma; \mathbb{R}^{d \times d})$ if

$$\text{tr}(aD^2u) \in C_{\text{loc}}(\mathcal{O} \cup \Sigma) \quad \text{and} \quad \text{tr}(aD^2u) = 0 \quad \text{on } \Sigma,$$

and A in (1.3) has the *weak maximum principle property on $\mathcal{O} \cup \Sigma$* for this class of functions if

$$\begin{cases} Au \leq 0 \text{ on } \mathcal{O}, \\ u \leq 0 \text{ on } \partial\mathcal{O} \setminus \Sigma, \end{cases} \implies u \leq 0 \quad \text{on } \bar{\mathcal{O}}.$$

The innovation here, of course, lies in the nature of the boundary condition for u along Σ (see Remark 2.13 for a discussion of equivalent first-order boundary conditions). When \mathcal{O} is unbounded, the conditions on u are augmented by an assumption that $\sup_{\mathcal{O}} u < \infty$.

We shall find it very convenient to cleanly separate a discussion of conditions which may lead to A in (1.3) having the basic weak maximum principle property, which are provided in §5 from their applications for boundary value and obstacle problems which flow from this abstract property and which are discussed in sections 2, 3, and 4.

In §2, we consider applications of the weak maximum principle property to boundary value problems, including a comparison principle for C^2 subsolutions and supersolutions and uniqueness for C^2 solutions to the Dirichlet boundary problem (Proposition 2.16), a priori estimates for C^2 subsolutions, supersolutions, and solutions (Proposition 2.19), and an extension for the case of subsolutions which obey a growth condition on unbounded domains (Theorem 2.20).

Section 3 contains application of the weak maximum principle property to obstacle problems. We develop a theory of *continuous* supersolutions to the obstacle problem, modeled on the corresponding theory for the Dirichlet boundary value problem in [40, §2.8 & §6.3] for linear, second-order, uniformly elliptic partial differential equations on bounded domains. Our development has some independent interest since it is a theory of *classical* rather than viscosity solutions, unlike that of [16]. We then develop applications of the weak maximum principle property to $C^{1,1}$ solutions to obstacle problems, including a comparison principle for continuous supersolutions and

uniqueness for solutions to the obstacle problem (Theorem 3.15) and a weak maximum principle and a priori estimates for solutions and continuous supersolutions to the obstacle problem (Proposition 3.18).

In §4, we prove a strong maximum principle and develop its applications to boundary value problems with Neumann boundary conditions. We first prove a Hopf boundary point lemma (see Lemma 4.1) for operators A in (1.3) which may become degenerate on $\partial\mathcal{O}$ using a novel choice of barrier functions. We then apply our version of the Hopf lemma to prove a strong maximum principle suitable for operators with non-negative characteristic form (Theorem 4.6) and corresponding uniqueness results for solutions to equations with Neumann boundary conditions along $\partial\mathcal{O} \setminus \Sigma$ (Theorem 4.9 and Corollary 4.12).

Finally, in §5, we establish specific conditions on the coefficients (a, b, c) which ensure that the operator A in (1.3) has the weak maximum principle property on $\mathcal{O} \cup \Sigma$, initially for bounded C^2 functions on bounded domains (Theorem 5.1), and then for bounded C^2 functions on unbounded domains (Theorem 5.3).

1.3.2. Weak maximum principles for bilinear maps and operators on functions in Sobolev spaces and applications to variational equations and inequalities. We next consider variational equations and inequalities defined by bilinear maps, \mathfrak{a} , on weighted Sobolev spaces, $H^1(\mathcal{O}, \mathfrak{w})$, defined by weight, $\mathfrak{w} \in C(\mathcal{O}) \cap L^1(\mathcal{O})$ with $\mathfrak{w} > 0$ on \mathcal{O} , and a degeneracy coefficient, $\vartheta \in C_{\text{loc}}(\mathcal{O})$ with $\vartheta > 0$ on \mathcal{O} , where a Borel measurable function, u on \mathcal{O} , lies in $H^1(\mathcal{O}, \mathfrak{w})$ if

$$\int_{\mathcal{O}} (\vartheta |Du|^2 + (1 + \vartheta)u^2) \mathfrak{w} \, dx < \infty.$$

Given a non-empty open subset $\Sigma \subseteq \partial\mathcal{O}$ (which may be arbitrary for now), we say that \mathfrak{a} obeys the *weak maximum principle property on $\mathcal{O} \cup \Sigma$* for functions $u \in H^1(\mathcal{O}, \mathfrak{w})$ if

$$\begin{cases} \mathfrak{a}(u, v) \leq 0, \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}) \text{ with } v \geq 0 \text{ a.e. on } \mathcal{O}, \\ u \leq 0 \text{ on } \partial\mathcal{O} \setminus \Sigma \text{ in the sense of } H^1(\mathcal{O}, \mathfrak{w}) \end{cases} \implies u \leq 0 \text{ a.e. on } \mathcal{O},$$

where $H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ is the closure of $C_0^\infty(\mathcal{O} \cup \Sigma)$ in $H^1(\mathcal{O}, \mathfrak{w})$ and $C_0^\infty(\mathcal{O} \cup \Sigma) \subset C^\infty(\mathcal{O})$ is the linear subspace of smooth functions which have compact support in $\mathcal{O} \cup \Sigma$, and $u \leq 0$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$ if $u^+ \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$, where $u^+ = \max\{u, 0\}$. When \mathcal{O} is unbounded, the conditions on u are augmented by an assumption that $\text{ess sup}_{\mathcal{O}} u < \infty$.

In §6, we consider applications of the weak maximum principle property, including the comparison principle (Proposition 6.7) and a priori estimates (Proposition 6.10) for $H^1(\mathcal{O}, \mathfrak{w})$ supersolutions and solutions to variational equations. We also show that when a bilinear map \mathfrak{a} on $H^1(\mathcal{O}, \mathfrak{w})$ has a weak maximum principle property for subsolutions (on unbounded domains) which are bounded above, the property may extend to subsolutions which instead obey a growth condition (Theorem 6.16).

Section 7 contains applications of the weak maximum principle property to variational inequalities. We prove uniqueness for solutions to variational inequalities defined by bilinear maps \mathfrak{a} on $H^1(\mathcal{O}, \mathfrak{w})$, a comparison principle for supersolutions and uniqueness for solutions to variational inequalities (Theorem 7.2), and a priori estimates (Proposition 7.9) for $H^1(\mathcal{O}, \mathfrak{w})$ supersolutions and solutions to variational inequalities.

Finally, in §8, we prove the weak maximum principle for functions in weighted Sobolev spaces. We first show that a bilinear map of the form,

$$\mathfrak{a}(u, v) = \int_{\mathcal{O}} (a^{ij} u_{x_i} v_{x_j} + d^j u v_{x_j} - b^i u_{x_i} v + c u v) \mathfrak{w} \, dx, \quad \forall u, v \in C_0^\infty(\bar{\mathcal{O}}),$$

has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ (Theorem 8.11) when \mathcal{O} is bounded and its coefficients obey, for some positive constant, K , a.e. on \mathcal{O} and all $\eta \in \mathbb{R}^d$,

$$\begin{aligned} \vartheta|\eta|^2 &\leq \langle a\eta, \eta \rangle \leq K\vartheta|\eta|^2, \\ \langle d, \eta \rangle &\leq K\vartheta|\eta|, \\ \langle b, \eta \rangle &\leq K\vartheta|\eta|, \\ |c| &\leq K\vartheta, \end{aligned}$$

together with a non-negativity condition,

$$\mathfrak{a}(1, v) \geq 0, \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}) \text{ with } v \geq 0 \text{ a.e. on } \mathcal{O},$$

and a key *weighted Sobolev inequality* holds: given constants $1 \leq p \leq q < \infty$, there is a positive constant C such that, for any $u \in L^q(\mathcal{O}, \mathfrak{w})$ with $Du \in L^p(\mathcal{O}, \vartheta\mathfrak{w}; \mathbb{R}^d)$, one has

$$\|u\|_{L^q(\mathcal{O}, \mathfrak{w})} \leq C\|Du\|_{L^p(\mathcal{O}, \vartheta\mathfrak{w})}.$$

Given additional constraints on the coefficients, we then extend our weak maximum principle to the case of bounded $H^1(\mathcal{O}, \mathfrak{w})$ functions on unbounded domains (Theorem 8.15).

Our previous definition of the degeneracy locus, Σ , for $a(x)$ is not meaningful when the function $a : \mathcal{O} \rightarrow \mathbb{R}^{d \times d}$ is only Borel measurable but, with the aid of the degeneracy coefficient, ϑ , it may be reinterpreted as

$$\Sigma = \text{int}(\{x \in \partial\mathcal{O} : \vartheta(x) = 0\}),$$

and, we then obtain the required weighted Sobolev inequality (Corollary 8.6) for specific choices of $\vartheta(x)$, including a power of the distance, $\text{dist}(x, \Sigma)$, from $x \in \mathcal{O}$ to the degeneracy locus, $\Sigma \subseteq \partial\mathcal{O}$ via a result of Koch [47].

Several appendices contain additional results and comments which shed further light on our conclusions. Appendix A provides examples illustrating when the weak maximum principle holds for functions obeying suitable growth conditions on unbounded domains. Appendix B contrasts the weighted Sobolev inequality (Corollary 8.6) due to Koch with one due to V. Maz'ya. In appendix C, we compare the weak maximum principles and uniqueness theorems provided by our article with those of Fichera, Oleřnik, and Radkevič [63] in the case of the elliptic Heston operator, A , in Example 1.2 on a domain $\mathcal{O} \subseteq \mathbb{H}$ and show that those of Fichera, Oleřnik, and Radkevič are strictly weaker.

1.4. Comparison with previous research. A detailed comparison between our weak maximum principle for classical subsolutions (Theorems 5.1 and 5.3) and that of Fichera [34, 35, 58, 63] is provided in Appendix C in the case of Example 1.2. To describe one of the principal differences between the two weak maximum principles, we recall the definition of the *Fichera function* [63, Equations (1.1.2) & (1.1.3)] (taking into account our sign convention in (1.3) for the coefficients (a, b, c) of the non-divergence-form operator A),

$$b_0 = \left(b^k - a_{x_j}^{kj}\right) n_k \quad \text{on } \partial\mathcal{O},$$

where \vec{n} denotes *inward*-pointing unit normal vector field along $\partial\mathcal{O}$ (now assumed, for example, to be C^1). By using the Fichera weak maximum principle to decide whether to impose a Dirichlet condition for u along Σ to achieve uniqueness of solutions to (1.1) or (1.2), one finds that a Dirichlet boundary condition is required for u on Σ when $b_0 < 0$ on Σ , in addition to the usual Dirichlet condition on $\partial\mathcal{O} \setminus \Sigma$, whereas only a Dirichlet boundary condition for u on $\partial\mathcal{O} \setminus \Sigma$ is required when $b_0 \geq 0$ on Σ .

However, by instead imposing a Dirichlet boundary condition for u on $\partial\mathcal{O}\setminus\Sigma$ and a second-order boundary condition for $u \in C^2(\mathcal{O}) \cap C^1_{\text{loc}}(\mathcal{O} \cup \Sigma) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ on Σ , namely $\text{tr}(aD^2u) \in C_{\text{loc}}(\mathcal{O} \cup \Sigma)$ and $\text{tr}(aD^2u) = 0$ on Σ , we achieve uniqueness *independent* of the sign of the Fichera function, b_0 , on Σ . Since the second-order boundary condition is automatically obeyed by, for example, functions in the weighted Hölder spaces introduced by Daskalopoulos, Hamilton, and Koch [19, 47], we are naturally led to a more convenient and powerful framework for establishing existence, uniqueness, and regularity of solutions.

Similar remarks apply in the case of solutions to variational equations and inequalities and the framework of weighted Sobolev spaces introduced by Daskalopoulos and the author in [18] and by Koch in [47].

Stochastic representation (or Feynman-Kac) formulae provide a different approach to establishing uniqueness of solutions to (1.1) or (1.2) and, in the case of A as in Example 1.2, are obtained by Pop and the author in [32].

Uniqueness results for solutions to the parabolic porous medium equation (and its linearization) were established by Daskalopoulos, Hamilton, Rhee, and Koch in [19, 20, 47] but, unlike the linearization of the porous medium, for the coefficients (a, b) in (1.3) we permit $a(x)$ and $x \cdot b(x)$ to have quadratic growth with x as $x \rightarrow \infty$ and, even when the coefficients b^i are constant, we do not require that $b^i = 0$ when $i = 1, \dots, d-1$. A weak maximum principle for the parabolic (model) Kimura diffusion operator is given by Epstein and Mazzeo in [27, Proposition 4.1.1], who also employ a form of second-order boundary condition, together with a Hopf lemma and a strong maximum principle in [27, Lemma 4.2.4 & 4.2.5]. Related uniqueness results and weak maximum principles for classical (sub-)solutions to second-order, linear, degenerate elliptic and parabolic operators are proved by Pozio, Punzo, and Tesi in [59, 61, 62], but they do not make use of second-order boundary conditions.

A weak maximum principle for variational (sub-)solutions to second-order, nonlinear, degenerate elliptic operators is established by Bonafede [10], with a complex collection of hypotheses, while a weak maximum principle for variational (sub-)solutions to (1.1) is proved by Monticelli and Payne [54] when it is assumed that the coefficient matrix, $a(x)$, of A in (1.3) possesses a uniformly elliptic direction and when \mathcal{O} is bounded (conditions, among others, which we do not impose in this article). Borsuk obtains a weak maximum principle, a Hopf lemma, and a strong maximum principle [11, Theorem 6.2.1, Lemma 6.2.1 & Theorem 6.2.2] for (sub-)solutions to variational equations defined by certain degenerate elliptic quasilinear operators on non-smooth domains using weighted Sobolev spaces. Our maximum principle for variational (sub-)solutions to (1.1) (Theorems 8.11 and 8.15) generalizes that of Trudinger [70, Theorem 1].

1.5. Notation and conventions. For $x, y \in \mathbb{R}$, we denote $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$, while $x^+ := x \vee 0$ and $x^- := -(x \wedge 0)$. We let $B(p, R) \subset \mathbb{R}^d$ denote the open ball with radius $R > 0$ and center $p \in \mathbb{R}^d$. For an integer $k \geq 0$, we let $C^k(\mathcal{O})$ denote the vector space of functions whose derivatives up to order k are continuous on \mathcal{O} and let $C^k(\bar{\mathcal{O}})$ denote the Banach space of functions whose derivatives up to order k are *uniformly continuous* and *bounded* on \mathcal{O} [3, §1.25 & §1.26]. If $T \subseteq \partial\mathcal{O}$ is a relatively open set, we let $C^k_{\text{loc}}(\mathcal{O} \cup T)$ denote the vector space of functions, u , such that, for any precompact open subset $U \Subset \mathcal{O} \cup T$, we have $u \in C^k(\bar{U})$. When we label a condition an *Assumption*, then it is considered to be universal and in effect throughout this article and so not referenced explicitly in theorem and similar statements; when we label a condition a *Hypothesis*, then it is only considered to be in effect when explicitly referenced.

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Part 1. Weak and strong maximum principles for operators on smooth functions and applications to boundary value and obstacle problems

In this part of our article (§2, §3, §4, and §5), we develop weak and strong maximum principles for operators on smooth functions and their applications to boundary value and obstacle problems.

2. APPLICATIONS OF THE WEAK MAXIMUM PRINCIPLE PROPERTY TO BOUNDARY VALUE PROBLEMS

We shall encounter many different situations (for example, bounded or unbounded domains $\mathcal{O} \subset \mathbb{R}^d$, bounded or unbounded functions u with prescribed growth, and so on) where a basic maximum principle holds for linear, second-order, partial differential operators A in (1.3) with non-negative characteristic form and acting on a linear subspace of functions in $C^2(\mathcal{O})$. In order to unify our treatment of applications, we find it useful to isolate a key “weak maximum principle property” (Definition 2.8) and then derive the consequences which necessarily follow in an essentially formal manner. In this section we consider applications to boundary value problems. After providing some technical conditions in §2.1, we proceed to the main applications in §2.2, namely a comparison principle for C^2 subsolutions and supersolutions and uniqueness for C^2 solutions to the Dirichlet boundary problem (Proposition 2.16) and a priori estimates for C^2 subsolutions, supersolutions, and solutions (Proposition 2.19). In §2.3, we show that when operator has the weak maximum principle property for subsolutions which are bounded above, the property may also hold subsolutions which instead obey a growth condition (Theorem 2.20).

2.1. Preliminaries. We introduce the

Definition 2.1 (Non-negative definite characteristic form and boundary degeneracy locus). Suppose $a = (a^{ij})$ is an $\mathbb{R}^{d \times d}$ -valued function on $\bar{\mathcal{O}}$ which defines a *non-negative definite characteristic form*, that is³

$$\langle a\eta, \eta \rangle \geq 0 \quad \text{on } \bar{\mathcal{O}}, \quad \forall \eta \in \mathbb{R}^d. \quad (2.1)$$

We call

$$\Sigma := \text{int}(\{x \in \partial\mathcal{O} : a(x) = 0\}) \subseteq \partial\mathcal{O} \quad (2.2)$$

the *boundary degeneracy locus* defined by the coefficient matrix $a = (a^{ij})$. \square

In this article, we normally assume that Σ in (2.2) is non-empty. Of course, when the matrix $a(x)$ in Definition 2.1 is symmetric for all $x \in \partial\mathcal{O}$, then the definition of Σ in (2.2) is equivalent to the definition

$$\Sigma^0 := \{x \in \partial\mathcal{O} : \langle a(x)\eta, \eta \rangle = 0, \forall \eta \in \mathbb{R}^d\},$$

used by Fichera [35], Oleřnik, and Radkevič [58], [63, p. 308] (where the whole domain boundary is denoted by Σ). As in [63, §1.1], we have the

Definition 2.2 (Linear, second-order, partial differential operator with non-negative definite characteristic form). We call A in (1.3) a linear, second-order, partial differential operator with *non-negative definite characteristic form on \mathcal{O}* if the coefficient matrix, a , obeys (2.1).

³We follow the definition of Oleřnik, and Radkevič [63, p. 308] rather than Tricomi [63, p. 298], which requires in addition that $\langle a\eta, \eta \rangle > 0$ on \mathcal{O} for all $\eta \in \mathbb{R}^d \setminus \{0\}$; however, in the applications we have in mind, the latter condition is always satisfied and the degeneracy is confined to a subset of $\partial\mathcal{O}$.

Definition 2.3 (Second-order boundary condition). Let $a : \bar{\mathcal{O}} \rightarrow \mathbb{R}^{d \times d}$ obey⁴

$$a \in C_{\text{loc}}(\mathcal{O} \cup \Sigma; \mathbb{R}^{d \times d}). \quad (2.3)$$

We say that $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeys a *second-order boundary condition* along Σ defined by $a \in C_{\text{loc}}(\bar{\mathcal{O}}; \mathbb{R}^{d \times d})$ via (2.2) if

$$Du \in C_{\text{loc}}(\mathcal{O} \cup \Sigma; \mathbb{R}^d), \quad (2.4)$$

$$\text{tr}(aD^2u) \in C_{\text{loc}}(\mathcal{O} \cup \Sigma), \quad (2.5)$$

$$\text{tr}(aD^2u) = 0 \quad \text{on } \Sigma. \quad (2.6)$$

Remark 2.4 (Second-order boundary conditions and Hölder spaces). For certain second-order differential operators with non-negative characteristic form, such as the linearization of the porous medium equation, the second-order boundary condition (2.6) is a property of functions in suitable Hölder spaces, $C_s^{2+\alpha}(\mathcal{O} \cup \Sigma)$, defined in [19] for functions on a domain $\mathcal{O} \subset \mathbb{R}^d$ equipped with a Riemannian metric $g^{ij} = a^{ij}$, where a is as in Definition 2.1 and we replace the usual Euclidean distance function on $\bar{\mathcal{O}}$ in the definition of standard Hölder spaces by the Riemannian distance function, s , defined by the metric a^{ij} . See [19, Proposition I.12.1], [30, Lemma 3.1] and [29] for further discussion. \square

Remark 2.5 (Ventcel boundary conditions and Feller semigroups). The second-order boundary condition in Definition 2.3 may be viewed as a special case of a generalized Ventcel boundary condition [68, §7.1]. Around 1950, Feller completely characterized the analytic structure of one-dimensional diffusion processes, giving an intrinsic representation of the infinitesimal generator $\mathcal{A} = -A$ of a one-dimensional diffusion process and determined all possible boundary conditions which describe the domain, $\mathcal{D}(\mathcal{A})$. (A parallel analysis from the point of view of Sturm-Liouville operators may be found in [72].) In 1959, Ventcel studied the problem of determining all possible boundary conditions for multi-dimensional diffusion processes and a generalization of Ventcel's results to Feller processes, by Taira and many others, is described in [68, Chapter 7]. However, while the Ventcel boundary condition may be *exogenous* (that is, defined by an auxiliary, degenerate second-order operator, L , which is distinct from A), the second-order boundary condition in Definition 2.3 are *endogenous* (that is, naturally induced by A). \square

Remark 2.6 (Relationship with definition of degenerate elliptic operators in the theory of viscosity solutions). Consider a second-order partial differential equation of the form

$$F(x, u, Du, D^2u) = 0, \quad \forall x \in \mathcal{O},$$

defined by a map

$$F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \rightarrow \mathbb{R},$$

where $\mathcal{S}^d \subset \mathbb{R}^{d \times d}$ denotes the subset of real, symmetric matrices. The map, F , is called *degenerate elliptic* (in the sense of viscosity solutions) [15, p. 1] if it is increasing with respect to its matrix argument,

$$F(x, r, \eta, X) \leq F(x, r, \eta, Y), \quad \forall X \leq Y \text{ and } (x, r, \eta) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d,$$

where $X \leq Y$ in the sense of \mathcal{S}^d if

$$\langle X\xi, \xi \rangle \leq \langle Y\xi, \xi \rangle, \quad \forall \xi \in \mathbb{R}^d.$$

⁴It would be more than sufficient for the application of Definition 2.3 in this article to require only that a is continuous on some open neighborhood of $\bar{\Sigma}$ in $\bar{\mathcal{O}}$.

When F is degenerate elliptic in the preceding sense, the map is called *proper* if

$$F(x, r, \eta, X) \leq F(x, s, \eta, Y), \quad \forall X \leq Y, r \leq s \text{ and } (x, \eta) \in \mathcal{O} \times \mathbb{R}^d.$$

Of course, the linear operators A in (1.3) and equations considered in this article will always define degenerate, proper maps $F(x, u, Du, D^2u) := Au - f$ since the coefficient matrix a of D^2u will obey (2.1) and the coefficient of u will obey $c \geq 0$ on \mathcal{O} ; however, the converse is not true — when F is degenerate elliptic in the sense of [15, p. 1], it does not imply that A is degenerate elliptic in the sense of this article (meaning a lack of strict inequality in (1.3) on $\bar{\mathcal{O}}$). \square

To simplify the statements of our definitions and results, we shall make the universal

Assumption 2.7 (Partial differential operator with non-negative characteristic form). Given a domain, $\mathcal{O} \subseteq \mathbb{R}^d$, the coefficients of the linear, second-order partial differential operator $A : C^2(\mathcal{O}) \rightarrow C(\mathcal{O})$ as in (1.3) obey (2.1), (2.3), and the open subset, $\Sigma \subseteq \partial\mathcal{O}$, is given by (2.2).

We can now state the key

Definition 2.8 (Weak maximum principle property for C^2 subsolutions). We say that an operator A in (1.3) obeys the *weak maximum principle property on $\mathcal{O} \cup \Sigma$* if whenever $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeys (2.4), (2.5), (2.6) and⁵

$$Au \leq 0 \quad \text{on } \mathcal{O}, \tag{2.7}$$

$$u \leq 0 \quad \text{on } \partial\mathcal{O} \setminus \Sigma, \tag{2.8}$$

and, if \mathcal{O} is unbounded, $\sup_{\mathcal{O}} u < \infty$, then

$$u \leq 0 \quad \text{on } \bar{\mathcal{O}}.$$

Examples of operators, A , with the weak maximum principle property on $\mathcal{O} \cup \Sigma$ are provided by Theorems 5.1, 5.3, 2.20, and (when $\Sigma = \emptyset$) [40, Theorem 3.1 & Corollary 3.2].

Remark 2.9 (Auxiliary boundedness condition in the definition of the weak maximum principle property). Once A is known to have the weak maximum principle property on $\mathcal{O} \cup \Sigma$, the condition that u be bounded above when \mathcal{O} is unbounded (which could be replaced by a growth condition via Theorem 2.20) plays no role in most applications of the weak maximum principle property. \square

Remark 2.10 (Auxiliary conditions required for linear, second-order elliptic differential operators to have the weak maximum principle property on $\mathcal{O} \cup \Sigma$). In order to prove that A has the weak maximum principle property on $\mathcal{O} \cup \Sigma$, we shall need to define open subset, $\Sigma \subseteq \partial\mathcal{O}$, via (2.2) and require that A to obey auxiliary conditions such as (2.1), (2.14), (2.13), (5.2) or (5.3), (5.4), (5.7) in order to prove that A has the weak maximum principle property on $\mathcal{O} \cup \Sigma$. However, once A is known to have the weak maximum principle property on $\mathcal{O} \cup \Sigma$, neither the definition of the open subset, $\Sigma \subseteq \partial\mathcal{O}$, via (2.2) nor these auxiliary conditions play any role in most applications of the weak maximum principle property. \square

Remark 2.11 (Role of the second-order boundary condition). We chose to make the auxiliary conditions (2.4), (2.5), (2.6) for u along Σ part of the definition of the function space for u and, as in [19, 30], they are properties of functions in $C_s^{2+\alpha}(\mathcal{O} \cup \Sigma)$. In the application of the weak maximum principle property, the second-order boundary condition plays no explicit role but we implicitly need to restrict functions $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ to *some* linear subspace where the replacement of $\partial\mathcal{O}$ by $\partial\mathcal{O} \setminus \Sigma$ makes sense and the subclass of functions obeying a second-order

⁵The condition (2.6) could be replaced by $\text{tr}(aD^2u) \leq 0$ on Σ , for a slight increase in generality, as is clear from Step 1 of the proof of Theorem 5.1.

boundary condition provides the arena in which we shall give examples of operators, A , having the weak maximum principle property. \square

Remark 2.12 (Non-negative zeroth-order coefficient). The condition (2.13) on the coefficient of u , namely,

$$c \geq 0 \quad \text{on } \mathcal{O}$$

is not explicitly required in Definition 2.8 since A is already assumed to possess the weak maximum principle property on $\mathcal{O} \cup \Sigma$, although simple counterexamples to the weak maximum principle exist when this condition is relaxed in general [40, p. 33]; the condition $c \geq 0$ on \mathcal{O} is a necessary condition for the strong maximum principle [60, Exercise 2.1]. \square

Remark 2.13 (Equivalent first-order boundary conditions). If $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}^1(\mathcal{O} \cup \Sigma) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeys (2.5) and (2.7), with A as in (1.3), then the second-order boundary condition (2.6) is equivalent to

$$-\langle b, Du \rangle + cu \leq 0 \quad \text{on } \Sigma. \quad (2.9)$$

Indeed, when we have equality in (2.7), and thus equality in (2.9), the condition (2.9) is analogous to the boundary condition proposed by Heston [45, Equation (9)] for the parabolic terminal/boundary problem corresponding to the boundary value problem (1.1): one obtains

$$-\langle b, Du \rangle + cu = f \quad \text{on } \Sigma, \quad (2.10)$$

for the boundary value problem (1.1) when f is non-zero and $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}^1(\mathcal{O} \cup \Sigma) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeys (2.5). Indeed, the parabolic version of the condition (2.10) (normally when $f = 0$) is often used in the numerical solution of parabolic boundary value or obstacle problems in mathematical finance [25, Equation (22.19)], [73, Equation (15)]. \square

2.2. Applications of the weak maximum principle property to boundary value problems.

We begin with the

Definition 2.14 (Classical solution, subsolution, and supersolution to a boundary value problem with Dirichlet data). Given functions $f \in C(\mathcal{O})$ and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, we call $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ a *subsolution* to a boundary value problem for an operator A in (1.3) with Dirichlet boundary condition along $\partial\mathcal{O} \setminus \Sigma$, if u obeys (2.4), (2.5), (2.6), $\sup_{\mathcal{O}} u < \infty$, and

$$Au \leq f \quad \text{on } \mathcal{O}, \quad (2.11)$$

$$u \leq g \quad \text{on } \mathcal{O} \setminus \Sigma. \quad (2.12)$$

We call u a *supersolution* if $-u$ is a subsolution and we call u a *solution* if it is both a subsolution and supersolution. \square

When $\Sigma = \emptyset$, then Definition 2.14 reverts to the standard definition of a solution to a boundary value problem [40]. The first application, of course, of the weak maximum principle property is to settle the question of *uniqueness* for solutions to the Dirichlet boundary problem.

Remark 2.15 (Auxiliary boundedness condition in the definition of subsolution, supersolution, and solution). The condition in Definition 2.14 that a subsolution, u , be bounded above when \mathcal{O} is unbounded can be replaced by a growth condition via Theorem 2.20 and similarly for a supersolution or solution. \square

Proposition 2.16 (Comparison principle for subsolutions and supersolutions and uniqueness for solutions to the Dirichlet boundary problem). *Let A in (1.3) obey the weak maximum principle property on $\mathcal{O} \cup \Sigma$. Let $f \in C(\mathcal{O})$ and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$. Suppose u_1 is a subsolution and u_2 a supersolution to the Dirichlet boundary problem in the sense of Definition 2.14. Then $u_2 \geq u_1$ on $\bar{\mathcal{O}}$ and, if u_1, u_2 are solutions, then $u_2 = u_1$ on $\bar{\mathcal{O}}$.*

Proof. Since u_1 is a subsolution and u_2 a supersolution, then $u_1 - u_2$ is a subsolution with $u_1 - u_2 = 0$ on $\partial\mathcal{O} \setminus \Sigma$ and thus

$$u_1 - u_2 \leq 0 \quad \text{on } \bar{\mathcal{O}},$$

because A has weak maximum principle property on $\mathcal{O} \cup \Sigma$. When u_1, u_2 are both solutions, then we also obtain $u_2 - u_1 \leq 0$ on $\bar{\mathcal{O}}$ and $u_2 = u_1$ on $\bar{\mathcal{O}}$. \square

We shall usually need to appeal to one of the following conditions.

Hypothesis 2.17 (Non-negative zeroth-order coefficient). The coefficient c in (1.3) obeys

$$c \geq 0 \quad \text{on } \mathcal{O}. \quad (2.13)$$

Hypothesis 2.18 (Uniform positive lower bound on zeroth-order coefficient). There is a constant $c_0 > 0$ such that the coefficient c in (1.3) obeys

$$c \geq c_0 \quad \text{on } \mathcal{O}. \quad (2.14)$$

We can now proceed to give the expected a priori estimates.

Proposition 2.19 (A priori estimates for C^2 subsolutions, supersolutions, and solutions). *Let A in (1.3) obey the weak maximum principle property on $\mathcal{O} \cup \Sigma$ and assume c obeys (2.13). Let $f \in C(\mathcal{O})$ and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$. Suppose u is a subsolution to the boundary value problem in the sense of Definition 2.14.*

- (1) *If $f \leq 0$ on \mathcal{O} and u is a subsolution for f and g , then*

$$u \leq 0 \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \bar{\mathcal{O}}.$$

- (2) *If f has arbitrary sign and u is a subsolution for f and g but, in addition, there is a constant $c_0 > 0$ such that c obeys (2.14), then*

$$u \leq 0 \vee \frac{1}{c_0} \sup_{\mathcal{O}} f \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \bar{\mathcal{O}}.$$

- (3) *If $f \geq 0$ on \mathcal{O} and u is a supersolution for f and g , then*

$$u \geq 0 \wedge \inf_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \bar{\mathcal{O}}.$$

- (4) *If f has arbitrary sign, u is a supersolution for f and g (Definition 2.14), and c obeys (2.14), then*

$$u \geq 0 \wedge \frac{1}{c_0} \inf_{\mathcal{O}} f \wedge \inf_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \bar{\mathcal{O}}.$$

- (5) *If $f = 0$ on \mathcal{O} and u is a solution for f and g (Definition 2.14), then*

$$\|u\|_{C(\bar{\mathcal{O}})} \leq \|g\|_{C(\overline{\partial\mathcal{O} \setminus \Sigma})}.$$

- (6) *If f has arbitrary sign, u is a solution for f and g , and c obeys (2.14), then*

$$\|u\|_{C(\bar{\mathcal{O}})} \leq \frac{1}{c_0} \|f\|_{C(\bar{\mathcal{O}})} \vee \|g\|_{C(\overline{\partial\mathcal{O} \setminus \Sigma})},$$

The terms $\sup_{\partial\mathcal{O} \setminus \Sigma} g$, and $\inf_{\partial\mathcal{O} \setminus \Sigma} g$, and $\|g\|_{C(\overline{\partial\mathcal{O} \setminus \Sigma})}$ in the preceding items are omitted when $\Sigma = \partial\mathcal{O}$.

The a priori estimate in Item (6) may be compared with [63, Theorem 1.1.2] (in the case of C^2 functions) and [63, Theorem 1.5.1 & 1.5.5] and [69, Lemma 2.8] (in the case of H^1 functions). However, because the coefficient matrix a of A in (1.3) is zero along $\Sigma \subseteq \partial\mathcal{O}$ in applications considered in this article, there is no analogue of [40, Theorem 3.7 & Corollary 3.8].

Proof of Proposition 2.19. For Items (1) and (2), we describe the proof when $\Sigma \subsetneq \partial\mathcal{O}$; the proof for the case $\Sigma = \partial\mathcal{O}$ is the same except that the suprema on the right-hand side are replaced by zero. When f has arbitrary sign, choose

$$M := 0 \vee \frac{1}{c_0} \sup_{\mathcal{O}} f \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g,$$

while if $f \leq 0$, choose

$$M := 0 \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g.$$

We may assume without loss of generality that $M < \infty$. Since $M \geq g$ on $\partial\mathcal{O} \setminus \Sigma$, we have

$$AM = cM \geq c_0 M \geq f \quad \text{on } \mathcal{O} \quad (\text{by (2.14)}),$$

when f has arbitrary sign and, when $f \leq 0$, we have

$$AM = cM \geq 0 \geq f \quad \text{on } \mathcal{O} \quad (\text{by (2.13)}).$$

Thus, M is a supersolution and so $u \leq M$ on $\bar{\mathcal{O}}$ by Proposition 2.16, which gives Items (1) and (2). Items (3) and (4) follow from Items (1) and (2) by noting that $-u$ is a subsolution for $-f$ and $-g$ if u is a supersolution for f and g . Item (5) follows by combining Items (1) and (3), while Item (6) follows by combining Items (2) and (4). \square

2.3. Applications of the weak maximum principle property to unbounded functions.

Our Definition 2.8 of the weak maximum principle property requires that the subsolution, u , on an unbounded domain is bounded above. However, if an operator has the weak maximum principle property for subsolutions which are bounded above, we obtain an extension for subsolutions which instead obey a growth condition.

Theorem 2.20 (Weak maximum principle for unbounded C^2 functions on unbounded domains). *Let A be an operator as in (1.3), let $\mathcal{O} \subseteq \mathbb{R}^d$ be a possibly unbounded domain with $\Sigma \subseteq \partial\mathcal{O}$ as in (2.2), and let $\varphi \in C^2(\bar{\mathcal{O}})$ obey $0 < \varphi \leq 1$ on $\bar{\mathcal{O}}$. Let*

$$Bv := -[A, \varphi](\varphi^{-1}v), \quad \forall v \in C^2(\mathcal{O}), \quad (2.15)$$

and suppose that the differential operator,

$$\hat{A} := (A + B)v, \quad \forall v \in C^2(\mathcal{O}), \quad (2.16)$$

has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ (Definition 2.8) for functions u on $\bar{\mathcal{O}}$ which are bounded above. Then A has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ for functions u on $\bar{\mathcal{O}}$ which obey the growth condition,

$$u \leq C(1 + \varphi^{-1}) \quad \text{on } \mathcal{O}. \quad (2.17)$$

Proof. Clearly, we have

$$\begin{aligned} \hat{A}(\varphi u) &= A\varphi u + B\varphi u \\ &= \varphi Au + [A, \varphi]u + B\varphi u \\ &\leq 0 \quad \text{on } \mathcal{O} \quad (\text{by (2.7) and (2.15)}). \end{aligned}$$

Since $\varphi u \leq C(\varphi + 1) \leq 2C$ on \mathcal{O} , then $\varphi u \leq 0$ on \mathcal{O} since \widehat{A} has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ for functions u on $\bar{\mathcal{O}}$ which are bounded above. Thus, $u \leq 0$ on $\bar{\mathcal{O}}$. \square

3. APPLICATIONS OF THE WEAK MAXIMUM PRINCIPLE PROPERTY TO OBSTACLE PROBLEMS

We now turn to the application of the weak maximum principle property to obstacle problems. The application is complicated by the fact that the optimal interior regularity of solutions to obstacle problems is $C^{1,1}(\mathcal{O})$ rather than $C^2(\mathcal{O})$. For this reason, we first consider in §3.1 a simple, if *artificial*, special case where the solutions are assumed to be C^2 , and prove a comparison principle for C^2 supersolutions and uniqueness for C^2 solutions to the obstacle problem (Proposition 3.3) and derive a priori estimates for C^2 supersolutions and C^2 solutions to obstacle problems (Proposition 3.5).

In §3.2, we develop a theory of *continuous* supersolutions to the obstacle problem, modeled on the corresponding theory for the Dirichlet boundary value problem in [40, §2.8 & §6.3] for linear, second-order, uniformly elliptic partial differential equations on bounded domains. Our development has some independent interest since it is a theory of *classical* rather than viscosity solutions, unlike that of [16]. We then consider the full problem of developing applications of the weak maximum principle property to $C^{1,1}(\mathcal{O})$ solutions to obstacle problems. In particular, we prove a comparison principle for continuous supersolutions and uniqueness for solutions to the obstacle problem (Theorem 3.15) and a weak maximum principle and a priori estimates for solutions and continuous supersolutions to the obstacle problem (Proposition 3.18).

3.1. Applications of the weak maximum principle property to obstacle problems with C^2 supersolutions. We have the

Definition 3.1 (Solution and C^2 supersolution to an obstacle problem). Given functions $f \in C(\mathcal{O})$, and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, and $\psi \in C^{1,1}(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeying $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$ and (2.4), (2.5), (2.6), we call $u \in C^{1,1}(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ a *solution* to an obstacle problem for an operator A in (1.3) with Dirichlet boundary condition along $\partial\mathcal{O} \setminus \Sigma$, if u obeys (2.4), (2.5), (2.6), $\sup_{\mathcal{O}} |u| < \infty$, and

$$\min\{Au - f, u - \psi\} = 0 \quad \text{a.e. on } \mathcal{O}, \quad (3.1)$$

$$u = g \quad \text{on } \mathcal{O} \setminus \Sigma. \quad (3.2)$$

We call $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ a C^2 *supersolution* to the obstacle problem if u obeys (2.4), (2.5), (2.6), $\sup_{\mathcal{O}} u > -\infty$, and (3.1) and (3.2) hold with $=$ replaced by \geq .

The comments in Remark 2.15 apply equally well to Definition 3.1.

Remark 3.2 (Regularity properties of the obstacle function). We require the obstacle, ψ , in Definition 3.1 to share the same regularity properties as those of the solution, u , since (unless the obstacle problem is trivial, in the sense that $u \geq \psi$ solves $Au = f$ on \mathcal{O}), the subset $\bar{\mathcal{O}} \cap \{u = \psi\}$ will be non-empty. However, there are many examples in applications where $\psi \in C_{\text{loc}}^{0,1}(\bar{\mathcal{O}})$ but $\psi = \psi_1 \wedge \psi_2$ and $\psi_1, \psi_2 \in C_{\text{loc}}^2(\bar{\mathcal{O}})$ are convex (such as the perpetual American put [66, §8.3]) and the obstacle problem in Definition 3.1 can still be well-posed in such cases.

Proposition 3.3 below for solutions to the obstacle problem is an analogue of Theorem 7.2, which applies to $H^1(\mathcal{O}, \mathfrak{w})$ solutions to a variational inequality, and Theorem 7.7, which applies to $H^2(\mathcal{O}, \mathfrak{w})$, that is, strong solutions to the obstacle problem. We may also compare Propositions 3.3 and 3.5 with [65, Theorems 4.5.1, 4.6.1, 4.6.6, & 4.7.4, and Corollary 4.5.2] for the case of variational inequalities.

Proposition 3.3 (Comparison principle for C^2 supersolutions and uniqueness for C^2 solutions to the obstacle problem). *Let A in (1.3) obey the weak maximum principle property on $\mathcal{O} \cup \Sigma$. Let $f \in C(\mathcal{O})$, and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, and $\psi \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ with $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$. Suppose u_1 is a solution and u_2 is a C^2 supersolution to the obstacle problem in the sense of Definition 3.1. Then $u_2 \geq u_1$ on $\bar{\mathcal{O}}$ and if u_2 is also a solution, then $u_2 = u_1$ on $\bar{\mathcal{O}}$.*

Proof. Suppose u_1 is a solution and u_2 a supersolution to the obstacle problem and that $\mathcal{U} := \mathcal{O} \cap \{u_1 > u_2\}$ is non-empty. Observe that $\partial\mathcal{U} = (\bar{\mathcal{O}} \cap \{u_1 = u_2\}) \cup (\partial\mathcal{O} \cap \{u_1 > u_2\})$ and because $u_1 = g \leq u_2$ on $\partial\mathcal{O} \setminus \Sigma$, we must have $\partial\mathcal{O} \cap \{u_1 > u_2\} = \Sigma \cap \{u_1 > u_2\}$. Therefore, $u_1 - u_2 \leq 0$ on $\partial\mathcal{U} \setminus \Sigma$. We have $u_1 \in C^2(\mathcal{U})$ by elliptic regularity [40, Theorem 6.13] and $u_2 \in C^2(\mathcal{U})$ because u_2 is a supersolution, so $u_1 - u_2 \in C^2(\mathcal{U})$ and $u_1 - u_2$ obeys (2.4), (2.5), (2.6) with \mathcal{U} in place of \mathcal{O} . Moreover, $A(u_1 - u_2) \leq 0$ on \mathcal{U} and so $u_1 - u_2 \leq 0$ on \mathcal{U} by the weak maximum principle property for A (Definition 2.8), contradicting our assertion that \mathcal{U} is non-empty. Hence, $u_1 \leq u_2$ on \mathcal{O} and thus $u_1 \leq u_2$ on $\bar{\mathcal{O}}$. \square

Remark 3.4 (Relaxation of the $C^2(\mathcal{O})$ regularity condition in Propositions 3.3 and 3.5). We include Propositions 3.3 and 3.5 because their proofs short, transparent, and insightful, but a far more useful versions are provided by Propositions 3.15 and 3.18, respectively, where the $C^2(\mathcal{O})$ restriction is removed.

We then have the

Proposition 3.5 (Weak maximum principle and a priori estimates for C^2 supersolutions and C^2 solutions to obstacle problems). *Let A in (1.3) obey the weak maximum principle property on $\mathcal{O} \cup \Sigma$ and assume c obeys (2.13). Let $f \in C(\mathcal{O})$, and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, and $\psi \in C_{\text{loc}}(\bar{\mathcal{O}})$ with $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$. Suppose u is a C^2 supersolution to the obstacle problem in the sense of Definition 3.1.*

- (1) *If $f \geq 0$ on \mathcal{O} , then*

$$u \geq 0 \wedge \inf_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \bar{\mathcal{O}}.$$

- (2) *If f has arbitrary sign but there is a constant $c_0 > 0$ such that c obeys (2.14), then*

$$u \geq 0 \wedge \frac{1}{c_0} \inf_{\mathcal{O}} f \wedge \inf_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \bar{\mathcal{O}}.$$

- (3) *If $f \leq 0$ on \mathcal{O} and u is a C^2 solution for f and g and ψ , then*

$$u \leq 0 \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g \vee \sup_{\mathcal{O}} \psi \quad \text{on } \bar{\mathcal{O}}.$$

- (4) *If f has arbitrary sign, u is a C^2 solution for f and g and ψ , and c obeys (2.14), then*

$$u \leq 0 \vee \frac{1}{c_0} \sup_{\mathcal{O}} f \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g \vee \sup_{\mathcal{O}} \psi \quad \text{on } \bar{\mathcal{O}}.$$

- (5) *If u_1 and u_2 are C^2 solutions, respectively, for $f_1 \geq f_2$ and $\psi_1 \geq \psi_2$ on \mathcal{O} , and $g_1 \geq g_2$ on $\partial\mathcal{O} \setminus \Sigma$, then*

$$u_1 \geq u_2 \quad \text{on } \bar{\mathcal{O}}.$$

- (6) *If u_1 and u_2 are C^2 solutions, respectively, for f_1, ψ_1 and f_2, ψ_2 on \mathcal{O} , and g_1 and g_2 on $\partial\mathcal{O} \setminus \Sigma$, and c obeys (2.14), then*

$$\|u_1 - u_2\|_{C(\bar{\mathcal{O}})} \leq \frac{1}{c_0} \|f_1 - f_2\|_{C(\bar{\mathcal{O}})} \vee \|g_1 - g_2\|_{C(\bar{\mathcal{O}} \setminus \Sigma)} \vee \|\psi_1 - \psi_2\|_{C(\bar{\mathcal{O}})},$$

while if $f_1 = f = f_2$ and c obeys (2.13), then

$$\|u_1 - u_2\|_{C(\bar{\mathcal{O}})} \leq \|g_1 - g_2\|_{C(\overline{\partial\mathcal{O}\setminus\Sigma})} \vee \|\psi_1 - \psi_2\|_{C(\bar{\mathcal{O}})}.$$

Proof. Consider Items (1) and (2). Since u is a supersolution to the obstacle problem in Definition 3.1, then it is also a supersolution to the boundary value problem in Definition 2.14 (where ψ plays no role) and so Items (1) and (2) here just restate Items (3) and (4) in Proposition 2.19.

Consider Items (3) and (4) here. When $f \leq 0$ on \mathcal{O} and A obeys (2.13), let

$$M := 0 \vee \sup_{\partial\mathcal{O}\setminus\Sigma} g \vee \sup_{\bar{\mathcal{O}}} \psi,$$

while if f has arbitrary sign and A obeys (2.14), let

$$M := 0 \vee \frac{1}{c_0} \sup_{\bar{\mathcal{O}}} f \vee \sup_{\partial\mathcal{O}\setminus\Sigma} g \vee \sup_{\bar{\mathcal{O}}} \psi.$$

We may assume without loss of generality that $M < \infty$. Then $M \geq \psi$ on $\bar{\mathcal{O}}$ and $M \geq g$ on $\partial\mathcal{O} \setminus \Sigma$, while

$$AM = cM \geq 0 \geq f \quad \text{on } \mathcal{O},$$

when $f \leq 0$ on \mathcal{O} and

$$AM = cM \geq c_0 M \geq f \quad \text{on } \mathcal{O},$$

when f has arbitrary sign. Hence, M is a supersolution and so Proposition 3.3 implies that $u \leq M$ on $\bar{\mathcal{O}}$, which establishes Items (3) and (4). For Item (5), observe that u_1 is a supersolution for the obstacle problem in Definition 3.1 given by f_2, g_2, ψ_2 and thus $u_1 \geq u_2$ on $\bar{\mathcal{O}}$ by Proposition 3.3. For Item (6), define

$$m := \frac{1}{c_0} \|f_1 - f_2\|_{C(\bar{\mathcal{O}})} \vee \|g_1 - g_2\|_{C(\overline{\partial\mathcal{O}\setminus\Sigma})} \vee \|\psi_1 - \psi_2\|_{C(\bar{\mathcal{O}})} \quad \text{and} \quad u := u_2 + m.$$

Then

$$Au = Au_2 + Am \geq f_2 + cm \geq f_2 + \sup_{\bar{\mathcal{O}}} (f_1 - f_2) \geq f_1 \quad \text{on } \mathcal{O},$$

while

$$u \geq \psi_2 + \sup_{\bar{\mathcal{O}}} (\psi_1 - \psi_2) \geq \psi_1 \quad \text{on } \mathcal{O},$$

and

$$u \geq g_2 + \sup_{\partial\mathcal{O}\setminus\Sigma} (g_1 - g_2) \geq g_1 \quad \text{on } \partial\mathcal{O} \setminus \Sigma.$$

Therefore, u is a supersolution for f_1, g_1, ψ_1 and so Proposition 3.3 implies that $u \geq u_1$ on $\bar{\mathcal{O}}$, and thus

$$u_1 - u_2 \leq m \quad \text{on } \mathcal{O}.$$

By interchanging the roles of u_1, u_2 in the preceding argument, the conclusion follows for the case $c \geq c_0 > 0$. For the case $c \geq 0$, we now define

$$m := \|\psi_1 - \psi_2\|_{C(\bar{\mathcal{O}})} \vee \|g_1 - g_2\|_{C(\overline{\partial\mathcal{O}\setminus\Sigma})} \quad \text{and} \quad u := u_2 + m,$$

so that

$$Au = Au_2 + Am \geq f + cm \geq f \quad \text{on } \mathcal{O},$$

and the remainder of the argument is identical. \square

3.2. Applications of the weak maximum principle property to obstacle problems with continuous supersolutions. Before introducing our definition of a continuous supersolution, we shall need the

Definition 3.6 (Local models for open neighborhoods of points in $\bar{\mathcal{O}}$). Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a domain with boundary, that $\partial\mathcal{O}$ is a C^2 submanifold-without-boundary in \mathbb{R}^d of dimension d , and that $\Sigma \subset \partial\mathcal{O}$ is a C^2 submanifold in \mathbb{R}^d of dimension $d-1$ with boundary $\partial\Sigma$ being a C^2 submanifold-without-boundary in \mathbb{R}^d of dimension $d-2$. We call $U \Subset \mathcal{O} \cup \Sigma$ a *model neighborhood* of a point $p \in \mathcal{O} \cup \Sigma$ if there is a C^2 -diffeomorphism $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\Phi(p) = 0$ and

- (1) If $p \in \mathcal{O}$, then $\Phi(U) = B(0, r)$, where

$$B(0, r) := \{x \in \mathbb{R}^d : |x| < r\}$$

is the open ball of radius $r > 0$;

- (2) If $p \in \Sigma$, then $\Phi(U) = B^+(0, r)$, where

$$B^+(0, r) := \{x \in \mathbb{R}^d : |x| < r, x_d > 0\} \subset \mathbb{R}^{d-1} \times \mathbb{R}_+$$

is the open half-ball, and

$$\Phi(\bar{U} \cap \Sigma) = B(0, r) \cap \{x_d = 0\};$$

- (3) If $p \in \partial\Sigma$, then $\Phi(U) = B^{++}(0, r)$, where

$$B^{++}(0, r) := \{x \in \mathbb{R}^d : |x| < r, x_{d-1} > 0, x_d > 0\} \subset \mathbb{R}^{d-2} \times \mathbb{R}_+ \times \mathbb{R}_+$$

is the open quarter-ball, and

$$\Phi(\bar{U} \cap (\partial\mathcal{O} \setminus \bar{\Sigma})) = B(0, r) \cap \{x_{d-1} = 0\} \cap \{x_d > 0\},$$

$$\Phi(\bar{U} \cap \Sigma) = B(0, r) \cap \{x_{d-1} > 0\} \cap \{x_d = 0\},$$

$$\Phi(\bar{U} \cap \partial\Sigma) = B(0, r) \cap \{x_{d-1} = x_d = 0\}.$$

We shall also need the

Hypothesis 3.7 (Existence and uniqueness of solutions to the Dirichlet problem on model neighborhoods). Suppose that $U \subset \mathbb{R}^d$ is a model neighborhood in the sense of Definition 3.9, that f and the coefficients of A in (1.3) belong to $C^\alpha(U)$ for some $\alpha \in (0, 1)$, that $\Sigma = \{x \in \partial U : a(x) = 0\}$, that $g \in C(\partial U \setminus \Sigma)$, and that A has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ in the sense of Definition 2.8.

Then there is a unique solution $u \in C^{2+\alpha}(U) \cap C(\bar{U})$ to the Dirichlet problem,

$$\begin{cases} Au = f & \text{on } U, \\ u = g & \text{on } \partial U \setminus \Sigma, \end{cases}$$

which obeys (2.4), (2.5), and (2.6).

Example 3.8 (Examples where the conclusions of Hypothesis 3.7 hold). When U is a ball and $\Sigma = \emptyset$ and $c \geq 0$ on U , then [40, Theorem 6.13] yields the conclusion of Hypothesis 3.7 for the resulting uniformly elliptic operator A on U . Although not a model neighborhood, since U in Hypothesis 3.7 is assumed bounded, the articles [19, 20, 30] provide examples of *parabolic* differential operators with non-negative characteristic form where the conclusion of Hypothesis 3.7 holds for $U = \mathbb{R}^{d-1} \times \mathbb{R}_+$, and $\Sigma = \mathbb{R}^{d-1} \times \{0\}$, and $b^d > 0$.

By analogy with the definitions of continuous superharmonic functions [40, §2.8], [44, Definition 6.1] or continuous supersolutions to linear, second-order elliptic partial differential equations [40, pp. 102–103], we have the⁶

Definition 3.9 (Supersolution to an obstacle problem). Let $\mathcal{O} \subseteq \mathbb{R}^d$ and $\Sigma \subseteq \partial\mathcal{O}$ be as in Definition 3.6 and let A in (1.3) obey Hypothesis 3.7. Given functions $f \in C^\alpha(\mathcal{O})$, and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, and $\psi \in C_{\text{loc}}(\bar{\mathcal{O}})$ obeying $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$, we call $v \in C(\mathcal{O})$ a *supersolution* to the obstacle problem (3.1), (3.2) if $v \geq \psi$ on \mathcal{O} and $\inf_{\mathcal{O}} v > -\infty$ and if for every open subset $U \Subset (\mathcal{O} \cup \Sigma) \cap \{v > \psi\}$ such that U is model neighborhood in the sense of Definition 3.6 and for every $\bar{v} \in C^2(U) \cap C(\bar{U})$ obeying

$$\begin{cases} A\bar{v} = f & \text{on } U, \\ v \geq \bar{v} & \text{on } \partial U \setminus \Sigma, \end{cases}$$

and (2.4), (2.5), (2.6), we then have

$$v \geq \bar{v} \quad \text{on } U.$$

We call $v \in C_{\text{loc}}(\bar{\mathcal{O}})$ a *supersolution relative to g on $\partial\mathcal{O} \setminus \Sigma$* if v is a supersolution and $v \geq g$ on $\partial\mathcal{O} \setminus \Sigma$. \square

Clearly, every solution or $C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ supersolution to the obstacle problem in the sense of Definition 3.1 is also a $C_{\text{loc}}(\bar{\mathcal{O}})$ supersolution relative to g . Note that \bar{v} in Definition 3.9 is *not* required to obey $\bar{v} \geq \psi$ on U and that, unlike in Definition 3.1, we only require that the obstacle, ψ , be continuous on $\bar{\mathcal{O}}$.

Remark 3.10 (Continuous supersolution and subsolution to a boundary value problem). When the obstacle, ψ , is omitted (in the obvious way) from Definition 3.9, then we call v a *supersolution (relative to g on $\partial\mathcal{O} \setminus \Sigma$)* to the boundary value problem (1.1). Similarly, we call v a *subsolution (relative to g on $\partial\mathcal{O} \setminus \Sigma$)* to the boundary value problem (1.1) if $-v$ is a supersolution (relative to g on $\partial\mathcal{O} \setminus \Sigma$).

Remark 3.11 (Perron’s method and solution to an obstacle or boundary value problem). It would be natural to seek a solution to the obstacle problem in Definition 3.1 via Perron’s method (see [40, Theorems 2.12 & 6.11]) and attempt to show that

$$u(x) = \inf_{v \in \mathcal{S}_{f,g,\psi}} v(x), \quad x \in \bar{\mathcal{O}}, \quad (3.3)$$

is a solution to the obstacle problem, where $\mathcal{S}_{f,g,\psi}$ is the set of all supersolutions to the obstacle problem in Definition 3.9 defined by f , g , and ψ . We shall not consider the question of existence itself in this article, beyond providing an analogue⁷ of [44, Lemma 6.3] in Lemma 3.12, but in Theorem 3.15, we show that if u is a solution then it can be expressed in the form (3.3). Naturally, this method would also apply to the boundary value problem (1.1).

Lemma 3.12 (Construction of smaller supersolutions to obstacle problems). *Let $\mathcal{O} \subseteq \mathbb{R}^d$ and $\Sigma \subseteq \partial\mathcal{O}$ be as in Definition 3.6 and let A obey Hypothesis 3.7. Let $f \in C^\alpha(\mathcal{O})$, and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, and $\psi \in C_{\text{loc}}(\bar{\mathcal{O}})$ obey $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$. Suppose that $v \in C(\mathcal{O})$ is a supersolution to the obstacle problem in the sense of Definition 3.9. Let $U \Subset (\mathcal{O} \cup \Sigma) \cap \{v > \psi\}$ be a model*

⁶While we could define supersolutions to the obstacle problem in the viscosity sense — see [16, Example 1.7 & Definition 2.2] and [51] — this simpler and more tractable definition will suffice for our applications.

⁷Lemma 6.3 in [44] asserts that if $v \in C(\bar{\mathcal{O}})$ is subharmonic, $B \Subset \mathcal{O}$, and w is defined by $\Delta w = 0$ on B and $w = v$ on $\bar{\mathcal{O}} \setminus B$, then w is subharmonic and $v \leq w$ on $\bar{\mathcal{O}}$.

neighborhood in the sense of Definition 3.6. Define $\bar{v} \in C(\mathcal{O})$ by setting $\bar{v} = v$ on $\mathcal{O} \setminus U$ and choosing $\bar{v} \in C^{2+\alpha}(U) \cap C(\bar{U})$ to be the unique solution to

$$\begin{cases} A\bar{v} = f & \text{on } U, \\ \bar{v} = v & \text{on } \partial U \setminus \Sigma. \end{cases}$$

If at least one of the following conditions hold,

- (1) $c \geq 0$ on U and $f \geq 0$ on U and $\psi \geq 0$ on U ;
- (2) $c \geq c_0 > 0$ on U for some positive constant c_0 ;

then \bar{v} is a supersolution for small enough $r > 0$ depending on $p \in \mathcal{O} \cup \Sigma$ and ψ .

Remark 3.13 (Application to Dirichlet boundary value problems). Lemma 3.12 also yields supersolutions to the Dirichlet boundary value problem (1.1).

Proof. Consider (1). Since $f \geq 0$ on U , then $A\bar{v} \geq 0$ and the weak maximum principle property implies that

$$\bar{v} \geq 0 \wedge \inf_{\partial U} v.$$

Since v is continuous on \mathcal{O} and $v(p) > \psi(p)$, we may choose r in Definition 3.6 small enough that

$$\inf_{\partial U} v \geq \sup_U \psi.$$

Since $\psi \geq 0$ on U and thus $v \geq 0$ on U , we obtain

$$\inf_U \bar{v} \geq \inf_{\partial U} v \geq \sup_U \psi,$$

and hence

$$\bar{v} \geq \psi \quad \text{on } \bar{U}.$$

But $\bar{v} = v \geq \psi$ on $\mathcal{O} \setminus \bar{U}$ and therefore \bar{v} is a supersolution.

Consider (2). In this case, f has arbitrary sign but $c \geq c_0 > 0$ and so we may choose a positive constant m such that $f + cm \geq f + c_0 m \geq 0$ on U and $\psi + m \geq 0$ on U . Then $A(\bar{v} + m) = A\bar{v} + cm = f + cm \geq 0$ on U and the conclusion for Item (1) yields $\bar{v} + m \geq \psi + m$ on U and thus we again obtain $\bar{v} \geq \psi$ on U , as desired. \square

We introduce an analogue of Definition 2.8.

Definition 3.14 (Strong maximum principle property for C^2 functions). Let $\mathcal{O} \subset \mathbb{R}^d$ be a domain and let $\Sigma \subseteq \partial\mathcal{O}$ be as in (2.2). We say that an operator A in (1.3) obeys the *strong maximum principle property* on $\mathcal{O} \cup \Sigma$ if whenever $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeys (2.4), (2.5) and (2.6) and $Au \leq 0$ on \mathcal{O} , then one of the following holds.

- (1) If $c = 0$ and u achieves a maximum in $\mathcal{O} \cup \Sigma$, then u is constant on $\bar{\mathcal{O}}$.
- (2) If $c \geq 0$ and u achieves a non-negative maximum in $\mathcal{O} \cup \Sigma$, then u is constant on $\bar{\mathcal{O}}$.

Theorem 4.6 and (when $\Sigma = \emptyset$) [40, Theorem 3.5] give examples of operators, A , with the strong maximum principle property.

Theorem 3.15 (Comparison principle for supersolutions and uniqueness for solutions to the obstacle problem). Let $\mathcal{O} \subseteq \mathbb{R}^d$ and $\Sigma \subseteq \partial\mathcal{O}$ be as in Definition 3.6 and let A in (1.3) obey Hypothesis 3.7 and have the strong maximum principle property on $\mathcal{O} \cup \Sigma$. Let $f \in C^\alpha(\mathcal{O})$, and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, and $\psi \in C_{\text{loc}}(\bar{\mathcal{O}})$ obey $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$. Suppose that u is a solution to the obstacle problem in the sense of Definition 3.1 and v a supersolution to the obstacle problem in

the sense of Definition 3.9. In addition, assume that each connected component C of $\mathcal{O} \cap \{u > \psi\}$ has the property that

$$\partial C \cap (\partial \mathcal{O} \setminus \Sigma \cap \{g > \psi\} \cup \{u = \psi\}) \neq \emptyset. \quad (3.4)$$

Then $u \leq v$ on $\bar{\mathcal{O}}$, and if v is also a solution, then $u = v$ on $\bar{\mathcal{O}}$.

We immediately obtain the following analogue of Proposition 2.16, where the hypothesis (3.4) is replaced by the simpler condition that $\Sigma \neq \partial \mathcal{O}$ and noting that the proof of Proposition 2.16 easily adapts to the case where u is a subsolution in the sense of Remark 3.10 rather than a solution in the sense of Definition 2.14.

Corollary 3.16 (Comparison principle for continuous supersolutions and uniqueness for solutions to the Dirichlet boundary value problem). *Assume the hypotheses of Theorem 3.15 on \mathcal{O} , Σ , A , f , and g . Suppose that u is a subsolution and v a supersolution to the boundary value problem (1.1) in the sense of Remark 3.10. In addition, assume that*

$$\partial \mathcal{O} \setminus \Sigma \neq \emptyset. \quad (3.5)$$

Then $u \leq v$ on $\bar{\mathcal{O}}$, and if u, v are both solutions, then $u = v$ on $\bar{\mathcal{O}}$.

Proof of Theorem 3.15. We may assume without loss of generality that $\mathcal{O} \cap \{u > \psi\}$ is non-empty, as otherwise we are done. Observe that $u = g \leq v$ on $\partial \mathcal{O} \setminus \Sigma$ and so $u - v \leq 0$ on $\partial \mathcal{O} \setminus \Sigma$, noting that, by hypothesis, $\partial \mathcal{O} \setminus \Sigma$ is non-empty. Let

$$M := \sup_{\mathcal{O}} (u - v),$$

noting that $M < \infty$ by hypothesis (Definitions 3.1 and 3.9), and consider

$$D := (\mathcal{O} \cup \Sigma) \cap \{u > \psi\} \cap \{u - v = M\}.$$

Clearly, D is a relatively closed subset of $(\mathcal{O} \cup \Sigma) \cap \{u > \psi\}$ in \mathbb{R}^d since u, v , and ψ are continuous on \mathcal{O} . We first assume that $(\mathcal{O} \cup \Sigma) \cap \{u > \psi\}$ is *connected* and aim to show that D is also a relatively open subset of $(\mathcal{O} \cup \Sigma) \cap \{u > \psi\}$.

We now proceed by adapting the proof⁸ of [44, Lemma 6.2]. Let $p \in \mathcal{O} \cup \Sigma$ and $U \Subset (\mathcal{O} \cup \Sigma) \cap \{u > \psi\}$ be a model neighborhood in the sense of Definition 3.6. By Hypothesis 3.7, there is a unique solution $\bar{v} \in C^{2+\alpha}(U) \cap C(\bar{U})$ to

$$\begin{cases} A\bar{v} = f & \text{on } U, \\ \bar{v} = v & \text{on } \partial U \setminus \Sigma. \end{cases}$$

Then $\bar{v} \leq v$ on U (recall that v is a supersolution) and hence

$$u - \bar{v} \geq u - v \quad \text{on } U.$$

Elliptic regularity, via Hypothesis 3.7, ensures that $u \in C^{2+\alpha}(U) \cap C(\bar{U})$ since $U \Subset (\mathcal{O} \cup \Sigma) \cap \{u > \psi\}$ and thus $Au = f$ on U . We have $A(u - \bar{v}) = Au - A\bar{v} = 0$ on U and $u - \bar{v} = u - v \leq M$ on ∂U , that is,

$$\begin{cases} A(u - \bar{v}) = 0 & \text{on } U, \\ u - \bar{v} \leq M & \text{on } \partial U \setminus \Sigma. \end{cases}$$

By the weak maximum principle property for A and the fact that $M \geq 0$, we must therefore have $u - \bar{v} \leq \sup_{\partial U \setminus \Sigma} (u - \bar{v})^+ \leq M$ on U ; in particular,

$$M \geq (u - \bar{v})(p) \geq (u - v)(p) = M.$$

⁸Lemma 6.2 in [44] asserts that if $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain, $u \in C(\bar{\mathcal{O}})$ is subharmonic, $v \in C(\bar{\mathcal{O}})$ is superharmonic, and $u \leq v$ on $\partial \mathcal{O}$, then $u \leq v$ on $\bar{\mathcal{O}}$.

Hence, $(u - \bar{v})(p) = M$ and so $u - \bar{v}$ has a maximum at $p \in U \cup \Sigma$. The strong maximum principle property for A implies that $u - \bar{v} \equiv M$ (constant) on U and thus $U \subset D$.

Since $\mathcal{O} \cap \{u > \psi\}$ is *connected* by our assumption and D is a relatively open and closed subset, then either $D = \emptyset$ or $D = \mathcal{O} \cap \{u > \psi\}$. If D is empty, then $u < v$ on $\mathcal{O} \cap \{u > \psi\}$ and $u \leq v$ on $\mathcal{O} \cap \{u = \psi\}$, so $u \leq v$ on \mathcal{O} and thus $u \leq v$ on $\bar{\mathcal{O}}$.

If $D = \mathcal{O} \cap \{u > \psi\}$, then $u - v$ is constant on $D = \mathcal{O} \cap \{u > \psi\}$ and thus $u - v$ is constant on $\bar{\mathcal{O}} \cap \{u > \psi\}$. We have

$$\partial(\mathcal{O} \cap \{u > \psi\}) = (\partial\mathcal{O} \cap \{u > \psi\}) \cup (\bar{\mathcal{O}} \cap \{u = \psi\}),$$

and

$$\begin{cases} u \leq g & \text{on } \partial\mathcal{O} \setminus \Sigma \cap \{u > \psi\}, \\ u \leq \psi & \text{on } \bar{\mathcal{O}} \cap \{u = \psi\}. \end{cases}$$

By hypothesis (3.4), at least one of the two preceding boundary portions is non-empty.

Case 1 (Boundary portion $\partial\mathcal{O} \setminus \Sigma \cap \{u > \psi\}$ is non-empty). Because $u \leq g \leq v$ on $\partial\mathcal{O} \setminus \Sigma$, then $u \leq v$ on $\partial\mathcal{O} \setminus \Sigma \cap \{u > \psi\}$ and we obtain $u \leq v$ on $\bar{\mathcal{O}} \cap \{u > \psi\}$. Thus, $u \leq v$ on $\bar{\mathcal{O}}$.

Case 2 (Boundary portion $\bar{\mathcal{O}} \cap \{u = \psi\}$ is non-empty). Because $u = \psi \leq v$ on $\bar{\mathcal{O}} \cap \{u = \psi\}$, then $u \leq v$ on $\partial(\mathcal{O} \cap \{u > \psi\})$ and we must have $u \leq v$ on $\bar{\mathcal{O}} \cap \{u > \psi\}$. Thus, $u \leq v$ on $\bar{\mathcal{O}}$.

Hence, when $\mathcal{O} \cap \{u > \psi\}$ is connected, we conclude that $u \leq v$ on $\bar{\mathcal{O}}$.

If $\mathcal{O} \cap \{u > \psi\}$ is *not connected*, we may apply the preceding argument to each connected component C of $\mathcal{O} \cap \{u > \psi\}$, noting that $\mathcal{O} \cap \{u > \psi\}$ is locally connected⁹ and so each connected component of $\mathcal{O} \cap \{u > \psi\}$, in particular C , is a relatively open and closed subset. We have (see Remark 3.17)

$$\partial C \subset \partial(\mathcal{O} \cap \{u > \psi\}) = (\partial\mathcal{O} \cap \{u > \psi\}) \cup (\bar{\mathcal{O}} \cap \{u = \psi\}),$$

and

$$\begin{cases} u \leq g & \text{on } \partial C \cap \partial\mathcal{O} \setminus \Sigma \cap \{u > \psi\}, \\ u \leq \psi & \text{on } \partial C \cap \bar{\mathcal{O}} \cap \{u = \psi\}. \end{cases}$$

We now appeal to hypothesis (3.4) and proceed exactly as before. This completes the proof. \square

Remark 3.17 (The boundary of a connected component C of a subset $X \subset \mathbb{R}^d$). This is a standard fact of point-set topology, but we include the proof for the sake of completeness. Recall that the maximal connected subsets (ordered by inclusion) of a nonempty subset $X \subset \mathbb{R}^d$ are the connected components of X and are non-empty. Let $C \subset X$ be a connected component and let $x \in \bar{C}$, where \bar{C} denotes the closure of C in \mathbb{R}^d . Suppose that $x \in \text{int}(X)$. Since \mathbb{R}^d is locally connected, there is a connected set $V \subset X$ which is an open neighborhood of x . Since C is connected and $V \cap C \neq \emptyset$, then $C \cup V \subset X$ is connected. Therefore, $V \subset C$. That is, $x \in \text{int}(C)$. Thus, for any $x \in C$, we have

$$x \in \text{int}(X) \implies x \in \text{int}(C).$$

Therefore,

$$\partial C = \bar{C} \setminus \text{int}(C) \subset \bar{C} \setminus \text{int}(X) \subset \bar{X} \setminus \text{int}(X) = \partial X.$$

This proof works for any locally connected topological space in place of \mathbb{R}^d .

We can now state and prove the desired version of Proposition 3.5, where the requirement that the solutions or supersolutions belong to $C^2(\mathcal{O})$ is removed.

⁹For each point $p \in \mathcal{O} \cap \{u > \psi\}$ and open subset $V \ni p$ and $V \subset \mathcal{O} \cap \{u > \psi\}$, there is a connected open subset U with $p \in U \subset V$ [55].

Proposition 3.18 (Weak maximum principle and a priori estimates for solutions and supersolutions to obstacle problems). *Let $\mathcal{O} \subseteq \mathbb{R}^d$ and $\Sigma \subseteq \partial\mathcal{O}$ be as in Definition 3.6 and let A in (1.3) obey Hypothesis 3.7 and have the strong maximum principle property on $\mathcal{O} \cup \Sigma$, and, in addition, require that c obeys (2.13). Let $f \in C^\alpha(\mathcal{O})$, and $g \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, and $\psi \in C_{\text{loc}}(\bar{\mathcal{O}})$ with $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$. Suppose u is a supersolution to the obstacle problem in the sense of Definition 3.9 and obeys (3.4).*

(1) *If $f \geq 0$ on \mathcal{O} , then*

$$u \geq 0 \wedge \inf_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \bar{\mathcal{O}}.$$

(2) *If f has arbitrary sign but there is a constant $c_0 > 0$ such that c obeys (2.14), then*

$$u \geq 0 \wedge \frac{1}{c_0} \inf_{\mathcal{O}} f \wedge \inf_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \bar{\mathcal{O}}.$$

(3) *If $f \leq 0$ on \mathcal{O} , and u is a solution for f and g and ψ (Definition 3.1), then*

$$u \leq 0 \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g \vee \sup_{\mathcal{O}} \psi \quad \text{on } \bar{\mathcal{O}}.$$

(4) *If f has arbitrary sign, u is a solution for f and g and ψ , and c obeys (2.14), then*

$$u \leq 0 \vee \frac{1}{c_0} \sup_{\mathcal{O}} f \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g \vee \sup_{\mathcal{O}} \psi \quad \text{on } \bar{\mathcal{O}}.$$

(5) *If u_1 and u_2 are solutions, respectively, for $f_1 \geq f_2$ and $\psi_1 \geq \psi_2$ on \mathcal{O} , and $g_1 \geq g_2$ on $\partial\mathcal{O} \setminus \Sigma$, and u_1 obeys (3.4), then*

$$u_1 \geq u_2 \quad \text{on } \bar{\mathcal{O}}.$$

(6) *If u_1 and u_2 are solutions, respectively, for f_1, ψ_1 and f_2, ψ_2 on \mathcal{O} , and g_1 and g_2 on $\partial\mathcal{O} \setminus \Sigma$, and c obeys (2.14), and u_1, u_2 obey (3.4), then*

$$\|u_1 - u_2\|_{C(\bar{\mathcal{O}})} \leq \frac{1}{c_0} \|f_1 - f_2\|_{C(\bar{\mathcal{O}})} \vee \|g_1 - g_2\|_{C(\bar{\partial\mathcal{O} \setminus \Sigma})} \vee \|\psi_1 - \psi_2\|_{C(\bar{\mathcal{O}})},$$

while if $f_1 = f = f_2$ and c just obeys (2.13), then

$$\|u_1 - u_2\|_{C(\bar{\mathcal{O}})} \leq \|g_1 - g_2\|_{C(\bar{\partial\mathcal{O} \setminus \Sigma})} \vee \|\psi_1 - \psi_2\|_{C(\bar{\mathcal{O}})}.$$

Proof. Consider Items (1) and (2). When $f \leq 0$ on \mathcal{O} , let

$$m := 0 \wedge \inf_{\partial\mathcal{O} \setminus \Sigma} g,$$

while if f has arbitrary sign and A obeys (2.14), let

$$m := 0 \wedge \frac{1}{c_0} \inf_{\mathcal{O}} f \wedge \inf_{\partial\mathcal{O} \setminus \Sigma} g.$$

We may assume without loss of generality that $m > -\infty$. Then $m \leq g$ on $\partial\mathcal{O} \setminus \Sigma$, while

$$Am = cm \leq 0 \leq f \quad \text{on } \mathcal{O},$$

when $f \leq 0$ on \mathcal{O} and

$$Am = cm \leq c_0 m \leq f \quad \text{on } \mathcal{O},$$

when f has arbitrary sign. Hence, m is a subsolution and u a supersolution to the boundary value problem (1.1) in the sense of Remark 3.10, and $m - u$ is bounded above on \mathcal{O} (by Definition 3.1). Thus, $m \leq u$ on $\bar{\mathcal{O}}$, as desired.

The proofs of Items (3), (4), (5), and (6) here are the same as the proofs of Items (3), (4), (5), and (6) in Proposition 3.5, except that appeals to Proposition 3.3 are now replaced by appeals to Theorem 3.15. \square

Remark 3.19 (A priori estimates for continuous supersolutions and subsolutions to a Dirichlet boundary value problem). Just as we extend Proposition 3.5 to Proposition 3.18 by allowing continuous rather than C^2 supersolutions, one could easily state and prove a version of Proposition 2.19 where continuous supersolutions and subsolutions in the sense of Remark 3.10 replace C^2 supersolutions and subsolutions in the sense of Definition 2.14.

4. STRONG MAXIMUM PRINCIPLE AND APPLICATIONS TO BOUNDARY VALUE PROBLEMS

The usual statements of the Hopf boundary point lemma [40, Lemma 3.4] require that A in (1.3) be uniformly elliptic on a domain, but a more careful analysis shows that it holds under much weaker hypotheses. We exploit our version of the Hopf lemma (see Lemma 4.1) to prove a strong maximum principle suitable for operators non-negative characteristic form (Theorem 4.6) and corresponding uniqueness results for solutions to equations with Neumann boundary conditions (Theorem 4.9 and Corollary 4.12).

4.1. A generalization of the Hopf boundary point lemma to linear, second-order differential operators with non-negative characteristic form. Noting carefully that our sign conventions for differential operators and normal vectors are *opposite* to those of [40] and that we do not distinguish in the statement of lemma between points $p_0 \in \partial\mathcal{O}$ where $a(p_0) = 0$ (that is, $p_0 \in \bar{\Sigma}$) or where $a(p_0)$ is positive definite (that is, $p_0 \in \partial\mathcal{O} \setminus \bar{\Sigma}$), we have

Lemma 4.1 (Hopf boundary point lemma for linear, second-order differential operators with non-negative characteristic form). *Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a domain and that \mathcal{O} obeys an interior sphere condition at $p_0 \in \partial\mathcal{O}$, with an open ball $B(p, R) \subset \mathcal{O}$ such that $p_0 \in \partial B(p, R)$. Require that the operator A in (1.3) obey (2.1) on $B(p, R)$, and*

$$c \geq 0 \quad \text{on } B(p, R), \quad (4.1)$$

and that at least one of the of the following conditions hold, where $h = \vec{n}(p_0)$ denotes the inward-pointing unit normal vector at p_0 :

$$\langle ah, h \rangle > 0 \quad \text{and} \quad \frac{\langle b, h \rangle}{\langle ah, h \rangle} \geq -2K_1 \quad \text{and} \quad \frac{c}{\langle ah, h \rangle} \leq K_0 \quad \text{on } B(p, R), \quad (4.2)$$

$$\text{or } c > 0 \quad \text{and} \quad \frac{\langle b, h \rangle}{c} \geq \nu_0 \quad \text{on } B(p, R), \quad (4.3)$$

for some non-negative constants, K_0, K_1 , or some positive constant, ν_0 , respectively.

Suppose that $u \in C^2(\mathcal{O})$ obeys (2.7), that is,

$$Au \leq 0 \quad \text{on } \mathcal{O}$$

and that u satisfies the conditions

- (i) u is continuous at p_0 ;
- (ii) $u(p_0) > u(p)$, for all $p \in \mathcal{O}$;
- (iii) $D_n u(p_0)$ exists,

where $D_n u(p_0)$ is the derivative of u at p_0 in the direction of the inward-pointing unit normal vector, $\vec{n}(p_0)$. Then the following hold.

(1) If $c = 0$ on \mathcal{O} , then $D_n u(p_0)$ obeys the strict inequality,

$$D_n u(p_0) < 0. \quad (4.4)$$

(2) If $c \geq 0$ on \mathcal{O} and $u(p_0) \geq 0$, then (4.4) holds.

(3) If $u(p_0) = 0$, then (4.4) holds irrespective of the sign of c .

Remark 4.2 (Comments on the Hopf lemma in the case of boundary points where the operator is degenerate). The Hopf lemma for points in $\{x_2 = 0\}$ for the Heston operator in Example 1.2 was proved independently by P. Daskalopoulos in an unpublished manuscript using a barrier function similar to that in the proof of [40, Lemma 3.4]. I am grateful to her for suggesting that such results should hold even at boundary points where the operator becomes degenerate. As pointed out to me by C. Pop, another version of the Hopf lemma for was obtained by Epstein and Mazzeo as [27, Lemma 4.2.4] for their generalized Kimura diffusion operators, but also proved using a barrier function similar to that in the proof of [40, Lemma 3.4].

The traditional proof of the Hopf lemma, as described by Gilbarg and Trudinger [40, Lemma 3.4], by Evans [28, §6.4.2], or by Han and Lin [44, Theorem 2.5], exploits the interior sphere condition by choosing the barrier function

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}, \quad x \in B(p, R) \setminus \bar{B}(p, \rho),$$

where $\alpha > 0$ is a constant which ultimately depends on bounds on the coefficients of A on the open annulus $B(p, R) \setminus \bar{B}(p, \rho)$, where $\rho \in (0, R)$ is a constant. This is the model for the barrier function chosen in the proof of [27, Lemma 4.2.4], but the resulting argument is quite difficult. As we shall see in our proof of Lemma 4.1, however, a choice of exponential-linear or linear barrier function instead will easily lead to the desired result.

That such a Hopf lemma should hold even at points in Σ can be seen by examining the *Kummer equation* [2, §13.1.1],

$$Au(x) := -xu_{xx}(x) - (b - x)u_x(x) + cu(x) = 0, \quad x \in \mathbb{R}_+,$$

where b and c are positive constants here. If $u_x(0)$ exists, then u is necessarily a constant multiple of a *confluent hypergeometric function of the first kind*, $u(x) = kM(x)$, by [2, Sections 13.1.2–4, 13.4.8, 13.4.21, and 13.5.5–10] and u is C^∞ on $[0, \infty)$ with $M(0) = 1$ and $u(0) = k \in \mathbb{R}$. (Indeed, when $a = c$ then $M(x) = e^x$.) The continuity of Au on \mathbb{R}_+ implies that $u_x(0) = kc/b$. If $k < 0$, then $Au = 0$ on \mathbb{R}_+ and u has a strict local maximum at $x = 0 \in \mathbb{R}_+$ and $u_x(0) < 0$, as predicted by Lemma 4.1.

Proof of Lemma 4.1. We use the strategy of the proof of [40, Lemma 3.4], with two choices of barrier functions, depending on whether condition (4.2) or (4.3) holds but both different from that in the proof of [40, Lemma 3.4] and an approach to exploiting the interior sphere condition which is also different from that in the proof of [40, Lemma 3.4].

Step 1 (Geometric set-up and application of a C^2 diffeomorphism). We may assume without loss of generality, using a translation of \mathbb{R}^d if needed, that $p_0 = 0 \in \mathbb{R}^d$ and $h = e_d$ and that $B(p, R)$ is contained in the open *upper* half-space $\{x_d > 0\}$. We now apply a C^2 diffeomorphism, $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with $\Phi(0) = 0$, to *flatten* the portion $\{0 \leq x_d < R\} \cap \partial B(p, R)$ of the boundary of $B(p, R)$ by pushing it *downward*, so that

$$\begin{aligned} \Phi(\{0 \leq x_d < R\} \cap \partial B(p, R)) &= \{x \in \mathbb{R}^d : |x| < R, x_d = 0\} \\ &= B(0, R) \cap \{x_d = 0\}. \end{aligned}$$

If $T \subset \partial\mathcal{O}$ is a small, relatively open neighborhood of $p_0 \in \partial\mathcal{O}$, the C^2 map Φ pushes $T \setminus \{p_0\}$ *downward* into the open *lower* half-space, $\{x_d < 0\}$:

$$\begin{aligned}\Phi(T \setminus \{p_0\}) &\subset \{x_d < 0\} \\ \Phi(\{x_d \geq R\} \cap \partial B(p, R)) &\Subset \Phi(\mathcal{O}).\end{aligned}$$

Henceforth, *after* applying the preceding diffeomorphism (and now denoting $\Phi(\mathcal{O})$ simply by \mathcal{O}), we may assume, without loss of generality, that the open *half-ball*

$$B^+(0, R) := \{x \in \mathbb{R}^d : |x| < R, x_d > 0\} \subset \mathcal{O},$$

has the property that

$$\{x_d > 0\} \cap \partial B^+(0, R) \Subset \mathcal{O}.$$

One can easily check (see the proofs of [28, Theorem 6.3.4], [40, Lemma 6.5 or Theorem 8.12], [48, Lemma 6.2.1] for similar arguments) that the conditions (4.2) and (4.3), thanks to (2.1), are invariant under the preceding coordinate changes, with the only modification being that the non-negative constants K_0, K_1 and positive constant ν_0 may be replaced by different non-negative constants or a positive constant, respectively, and the open ball, $B(p, R)$, in those conditions is replaced by the open half-ball, $B^+(0, R)$.

Step 2 (Construction of the barrier function when condition (4.2) holds). We choose

$$v(x) := e^{\alpha x_d} - 1, \quad x \in \mathbb{R}^d,$$

where $\alpha > 0$ is a constant yet to be determined. Clearly, $v(0) = 0$ and $v \geq 0$ on the half-space $\{x_d \geq 0\}$ and, in particular, $v \geq 0$ on $B^+(0, R)$. Moreover,

$$\begin{aligned}Av &= -\alpha^2 a^{dd} e^{\alpha x_d} - b^d \alpha e^{\alpha x_d} + c(e^{\alpha x_d} - 1) \\ &\leq -a^{dd} \left(\alpha^2 + \alpha \frac{b^d}{a^{dd}} - \frac{c}{a^{dd}} \right) e^{\alpha x_d} \\ &\leq -a^{dd} (\alpha^2 + 2K_1 \alpha - K_0) e^{\alpha x_d} \quad \text{on } B^+(0, R),\end{aligned}$$

using the hypothesis (4.1) to obtain $c \geq 0$ on $B^+(0, R)$ and noting that $a^{dd} > 0$ on $B^+(0, R)$ by hypothesis (4.2) and that a, b, c obey (4.2) (after applying a C^2 diffeomorphism) with $B(p, R)$ replaced by $B^+(0, R)$. But

$$\alpha^2 + 2K_1 \alpha - K_0 = (\alpha - K_1)^2 - K_1^2 - K_0 > 0$$

provided α obeys

$$\alpha > K_1 + \sqrt{K_0 + K_1^2}.$$

We fix such an α and thus obtain $Av < 0$ on $B^+(0, R)$.

Step 3 (Construction of the barrier function when condition (4.3) holds). We choose

$$v(x) := x_d, \quad \forall x \in \mathbb{R}^d,$$

and observe that

$$Av = -b^d v_{x_d} + cv = -b^d + cx_d.$$

We may assume without loss of generality, by decreasing R if needed, that $B^+(0, R) \subset \{x \in \mathbb{R}^d : 0 \leq x_d < \nu_0\}$ for ν_0 as in (4.3), and so the bound (4.3) (after applying a C^2 diffeomorphism) with $B(p, R)$ replaced by $B^+(0, R)$ yields

$$\inf_{B^+(0, R)} \frac{b^d}{c} \geq \nu_0 > 0.$$

Therefore,

$$Av = -c \left(\frac{b^d}{c} - x_d \right) \leq -c(\nu_0 - x_d) < 0 \quad \text{on } B^+(0, R),$$

and we again obtain $Av < 0$ on $B^+(0, R)$.

Step 4 (Verification that the weak maximum principle holds for A on $B^+(0, R)$). From Steps 2 and 3, we obtain $Av < 0$ on $B^+(0, R)$ for either choice of barrier function, v . Therefore, because A obeys (2.1) on $B^+(0, R)$ (note that this condition is invariant under the C^2 changes of coordinates employed here), the proof of [40, Theorem 3.1], with the role of $e^{\gamma x_1}$ in [40, p. 32] replaced by v , shows that the conclusions of [40, Theorem 3.1 & Corollary 3.2] hold for A on $B^+(0, R)$.

Step 5 (Application of the weak maximum principle). Since $u - u(0) < 0$ on \mathcal{O} and $u \in C_{\text{loc}}(\bar{\mathcal{O}})$ and $\{x_d > 0\} \cap \partial B^+(0, R) \subseteq \mathcal{O}$, we obtain

$$u(x) - u(0) \leq -m_0 < 0, \quad \forall x \in \{x_d > 0\} \cap \partial B^+(0, R),$$

for some positive constant, m_0 , depending on R and u . If (4.2) holds, then

$$v(x) = e^{\alpha x_d} - 1 \leq e^{\alpha R} - 1, \quad \forall x \in \{x_d > 0\} \cap \partial B^+(0, R),$$

while if (4.3) holds, then

$$v(x) = x_d \leq R, \quad \forall x \in \{x_d > 0\} \cap \partial B^+(0, R).$$

Hence, recalling that $R > 0$, there is a positive constant $m_1 := (e^{\alpha R} - 1) \vee R$ such that

$$v(x) \leq m_1, \quad \forall x \in \{x_d > 0\} \cap \partial B^+(0, R).$$

Consequently,

$$u(x) - u(0) + \varepsilon v(x) \leq -m_0 + \varepsilon m_1 \leq 0, \quad \forall x \in \{x_d > 0\} \cap \partial B^+(0, R),$$

provided we fix ε in the range $0 < \varepsilon \leq m_0/m_1$, while

$$u(x) - u(0) + \varepsilon v(x) = u(x) - u(0) \leq 0, \quad \forall x \in \{x_d = 0\} \cap \partial B^+(0, R),$$

since for either choice of barrier function we have $v(x) = 0$ when $x_d = 0$ and our hypothesis (ii) (with $p_0 = 0$) implies that $u(x) \leq u(0)$ on $\partial B^+(0, R) \subset \mathcal{O} \cup \{0\}$. But

$$A(u - u(0) + \varepsilon v) = Au - cu(0) + \varepsilon Av \leq -cu(0) \leq 0 \quad \text{on } B^+(0, R),$$

where the last inequality holds if $c = 0$ (as in Conclusion (1)), or $c \geq 0$ and $u(0) \geq 0$ (as in Conclusion (2)), or $u(0) = 0$ (as in Conclusion (3)). (For the case $u(0) = 0$, we simply note as in the proof of [40, Lemma 3.4] that we can replace A by $A + c^-$, where we write $c = c^+ - c^-$.)

The weak maximum principle therefore yields

$$u - u(0) + \varepsilon v \leq 0 \quad \text{on } B^+(0, R),$$

by virtue of Step 4.

Step 6 (Sign of the directional derivative of the subsolution at the boundary). We have

$$\frac{u(x) - u(0)}{x_d} \leq -\varepsilon \frac{v(x)}{x_d} = -\varepsilon \frac{v(x) - v(0)}{x_d}, \quad \forall x \in B^+(0, R),$$

If $v(x) = e^{\alpha x_d} - 1$, we have $v_{x_d} = \alpha e^{\alpha x_d}$ and $v_{x_d}(0) = \alpha > 0$, while if $v(x) = x_d$, we have $v_{x_d} = 1$. Taking the limit as $x_d \downarrow 0$ and noting that

$$v_{x_d}(0) = \begin{cases} \alpha & \text{if } a, b, c \text{ obey (4.2),} \\ 1 & \text{if } b, c \text{ obey (4.3),} \end{cases}$$

yields $v_{x_d}(0) \geq \alpha \wedge 1$ and

$$u_{x_d}(0) \leq -\varepsilon v_{x_d}(0) \leq -\varepsilon(\alpha \wedge 1) < 0,$$

and thus (4.4) holds.

This completes the proof. \square

Remark 4.3 (Application to the elliptic Heston operator). The hypotheses of Lemma 4.1 on the coefficients of A are obeyed in the case of the elliptic Heston operator, Example 1.2, where $d = 2$ and $\mathcal{O} \subset \mathbb{H}$ and $\Sigma = \text{int}(\partial\mathcal{O} \cap \partial\mathbb{H})$. For example, if $p_0 \in \Sigma$ then $h = \vec{n}(p_0) = e_2$, while $a^{22} = \sigma^2 y/2$ and $b^2 = \kappa(\theta - y)$, so

$$\frac{b^2}{a^{22}} = \frac{2\kappa(\theta - y)}{\sigma^2 y} \geq -\frac{2\kappa}{\sigma^2}, \quad \forall y \geq 0.$$

Thus, condition (4.2) is obeyed when $r = 0$, noting that $c = r$, while if $r > 0$, then

$$\frac{b^2}{c} = \frac{\kappa(\theta - y)}{r} \geq \frac{\kappa\theta}{2r}, \quad 0 \leq y < \theta/2,$$

and thus condition (4.3) is obeyed. \square

Remark 4.4 (Application to the linear, second-order, uniformly elliptic operators). Given a domain $\mathcal{O} \subset \mathbb{R}^d$, if A in (1.3) obeys (2.1), one has

$$0 \leq \lambda(x) \leq \langle a(x)\eta, \eta \rangle \leq \Lambda(x), \quad \forall x \in \mathcal{O}, \eta \in \mathbb{R}^d \setminus \{0\},$$

where $\lambda(x)$ and $\Lambda(x)$ are the minimum and maximum eigenvalues of $a(x)$. Then, A is *elliptic* on \mathcal{O} in the sense of [40, p. 31] if $\lambda > 0$ on \mathcal{O} , and *uniformly elliptic* on \mathcal{O} [40, p. 31] if

$$\sup_{\mathcal{O}} \left| \frac{\Lambda}{\lambda} \right| < \infty, \tag{4.5}$$

while A is *strictly elliptic* on \mathcal{O} [40, p. 31] if

$$\inf_{\mathcal{O}} \lambda > 0. \tag{4.6}$$

(The terminology is not universal — for example, compare [28, §6.1.1] — but we shall follow the convention of [40] in this article.) The hypotheses of [40, Lemma 3.4] require that A is uniformly elliptic on \mathcal{O} and that (see [40, Equation (3.2)])

$$\sup_{\mathcal{O}} \left| \frac{b}{\lambda} \right| < \infty \quad \text{and} \quad \sup_{\mathcal{O}} \left| \frac{c}{\lambda} \right| < \infty. \tag{4.7}$$

The bounds in (4.7), together with an assumption that A is elliptic on \mathcal{O} , ensure that the inequalities (4.2) hold and hence our Lemma 4.1 implies [40, Lemma 3.4], although the converse is not true, as illustrated in Remark 4.3. \square

Remark 4.5 (Hopf boundary point lemma for domains obeying an interior cone condition). The interior sphere condition can be relaxed in the classical Hopf lemma [40, Lemma 3.4], as noted in [40, p. 35 & p. 46], and generalizations to domains with non-smooth points are described in [52, 53, 56, 57]. We shall describe a generalization of Lemma 4.1 to such domains in [29]. \square

4.2. Strong maximum principle. By using Lemma 4.1 instead of [40, Lemma 3.4] in the proofs of [40, Theorem 3.5 & 3.6], we obtain the following statement of the strong maximum principle in Theorem 4.6. We provide details of the proof in order to emphasize the fact that the phrase “maximum in the interior of \mathcal{O} ” in our version of the strong maximum principle can be replaced by “maximum in Σ or the interior of \mathcal{O} ”, following the pattern observed by Daskalopoulos and Hamilton in [19] where Σ can often be treated as a set of “interior” points of the domain.

Theorem 4.6 (Strong maximum principle for C^2 functions). *Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a domain¹⁰. Require that the operator A in (1.3) obey (2.1), (2.3), (2.13), and that on each ball $B \subset \mathcal{O}$ with $\bar{B} \subset \mathcal{O} \cup \Sigma$, either (4.2) or (4.3) hold, where $\Sigma \subset \partial\mathcal{O}$ is as in (2.2). Assume, in addition, that Σ is C^1 and*

$$\langle b, \vec{n} \rangle > 0 \quad \text{on } \Sigma. \quad (4.8)$$

Suppose that $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeys (2.4), (2.5) and (2.6). If u obeys (2.7), that is,

$$Au \leq 0 \quad \text{on } \mathcal{O},$$

then the following hold.

- (1) *If $c = 0$ on \mathcal{O} and u attains a maximum in $\mathcal{O} \cup \Sigma$, then u is constant on $\bar{\mathcal{O}}$.*
- (2) *If $c \geq 0$ on \mathcal{O} and u attains a non-negative maximum in $\mathcal{O} \cup \Sigma$, then u is constant on $\bar{\mathcal{O}}$.*

Remark 4.7 (Strong maximum principle property). Theorem 4.6 implies that A has the strong maximum principle property, for C^2 functions, in the sense of Definition 3.14.

Proof of Theorem 4.6. Consider Conclusion (1). Assume, to the contrary, that u is non-constant on \mathcal{O} and achieves its maximum M on $\bar{\mathcal{O}}$ at a point in $\mathcal{O} \cup \Sigma$. Let $\mathcal{O}^- := \{x \in \mathcal{O} : u(x) < M\}$ and observe that \mathcal{O}^- is non-empty by our assumption that u is non-constant on \mathcal{O} . Let $p \in \mathcal{O}^-$ be such that $\text{dist}(p, \partial\mathcal{O}^-) < \text{dist}(p, \partial\mathcal{O} \setminus \Sigma)$ (if $\Sigma = \partial\mathcal{O}$, then any $p \in \mathcal{O}^-$ will do) and let $B \subset \mathcal{O}^-$ be the largest open ball centered at p and contained in \mathcal{O}^- . Then $u(p_0) = M$ for some $p_0 \in \partial B \cap \partial\mathcal{O}^-$ and $u < M$ on B . Note that $p_0 \in \mathcal{O} \cup \Sigma$, since $\text{dist}(p, \partial\mathcal{O}^-) < \text{dist}(p, \partial\mathcal{O} \setminus \Sigma)$ by construction (again, this condition is vacuous if $\Sigma = \partial\mathcal{O}$).

Case 1 ($p_0 \in \mathcal{O}$). We must have $Du(p_0) = 0$ since p_0 is an interior local maximum. However, by applying Conclusion (1) in Lemma 4.1 to the operator A on the domain B and boundary point $p_0 \in \partial B$, we obtain $Du(p_0) \neq 0$, a contradiction.

Case 2 ($p_0 \in \Sigma$). We may assume without loss of generality, by applying a C^2 diffeomorphism if needed, that $\mathcal{O} \cap B(p_0, r) = \mathbb{H} \cap B(p_0, r)$ for sufficiently small r , where $\mathbb{H} = \mathbb{R}^{d-1} \times \mathbb{R}_+$, so $\Sigma \cap B(p_0, r) = \{x_d = 0\} \cap B(p_0, r)$ and $\vec{n}(p_0) = e_d$. Using (2.6), and $b^d(p_0) > 0$ by (4.8), and $u_{x_i}(p_0) = 0$ for $1 \leq i \leq d-1$ since p_0 is a local maximum for u in Σ , and the fact that $c(p_0) \geq 0$ and $u(p_0) \geq 0$ by hypothesis, we would then have

$$Au(p_0) = -\text{tr}(aD^2u)(p_0) - \langle b(p_0), Du(p_0) \rangle + c(p_0)u(p_0) \geq -b^d(p_0)u_{x_d}(p_0) > 0,$$

where the final strict inequality follows from Conclusion (1) of Lemma 4.1, contradicting the fact that $Au \leq 0$ on \mathcal{O} by hypothesis and hence $Au \leq 0$ on $\mathcal{O} \cup \Sigma$ by (2.5).

Conclusion (2) follows by an identical argument when u achieves a non-negative maximum in $\mathcal{O} \cup \Sigma$, except that we now appeal to Conclusion (2) in Lemma 4.1. \square

Theorem 4.6 provides an interesting conclusion when $\Sigma = \partial\mathcal{O}$ and \mathcal{O} is bounded:

¹⁰Recall that by a “domain” in \mathbb{R}^d , we always mean a *connected*, open subset.

Corollary 4.8 (Strong maximum principle for C^2 functions on bounded domains and operators which are degenerate along the entire domain boundary). *Assume the hypotheses of Theorem 4.6 and, in addition, that $\Sigma = \partial\mathcal{O}$ and \mathcal{O} is bounded and $c = 0$. Then u is constant on $\bar{\mathcal{O}}$.*

Proof. Since $u \in C(\bar{\mathcal{O}})$ and u is bounded, then u necessarily achieves its maximum at some point in $\bar{\mathcal{O}} = \mathcal{O} \cup \Sigma$. But then Conclusion (1) in Theorem 4.6 implies that u must be constant on $\bar{\mathcal{O}}$. \square

We next consider the question of uniqueness in the Neumann problem and note here that it is important to distinguish between $\partial\mathcal{O}$ and $\partial\mathcal{O} \setminus \Sigma$.

Theorem 4.9 (Uniqueness for the Neumann problem for C^2 functions on bounded domains). *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain. Let A in (1.3) and $\Sigma \subsetneq \partial\mathcal{O}$ in (2.2) obey the hypotheses of Theorem 4.6 and assume that \mathcal{O} satisfies an interior sphere condition at each point of the boundary portion $\partial\mathcal{O} \setminus \Sigma$. Suppose that $u \in C^2(\mathcal{O}) \cap C(\bar{\mathcal{O}})$ obeys (2.4), (2.5), (2.6) and that*

$$Au = 0 \quad \text{on } \mathcal{O}.$$

If the derivative, $D_n u$, with respect to the inward-pointing normal vector field, \vec{n} on $\partial\mathcal{O} \setminus \Sigma$, is defined everywhere on $\partial\mathcal{O} \setminus \Sigma$ and

$$D_n u = 0 \quad \text{on } \partial\mathcal{O} \setminus \Sigma, \tag{4.9}$$

then u is identically constant on \mathcal{O} . In addition, if $c > 0$ at some point in \mathcal{O} , then $u \equiv 0$ on \mathcal{O} .

Proof. If u is not identically constant on \mathcal{O} , then either u or $-u$ achieves a non-negative maximum M on $\bar{\mathcal{O}}$. Since \mathcal{O} is bounded and $u \in C(\bar{\mathcal{O}})$, we may suppose that u achieves a non-negative maximum at some point $p_0 \in \bar{\mathcal{O}}$, as the argument when $-u$ achieves a non-negative maximum on $\bar{\mathcal{O}}$ will be identical. Therefore, $u(p_0) = M$ for some $p_0 \in \partial\mathcal{O} \setminus \Sigma$ since, because u is not identically constant on \mathcal{O} , Theorem 4.6 implies that $u < M$ on $\mathcal{O} \cup \Sigma$. But then Lemma 4.1 implies that $D_n u(p_0) < 0$, contradicting our hypothesis (4.9). Thus, we must have $u = M$, a constant, on $\bar{\mathcal{O}}$. If $c > 0$ at some point of $\bar{\mathcal{O}}$, the facts that $Au = cM$ and $Au = 0$ force $M = 0$. \square

Remark 4.10 (Interior sphere condition in the hypotheses of Theorem 4.9). In applications to mathematical finance, the hypothesis that \mathcal{O} obey an interior sphere condition at each point of the boundary portion $\partial\mathcal{O} \setminus \Sigma$ is not usually obeyed at points in $\partial\Sigma \subset \partial\mathcal{O} \setminus \Sigma$. Alternative hypotheses, more in harmony with applications, are discussed in [29].

Definition 4.11 (Classical solution to a boundary value problem with Neumann data). Let $\mathcal{O} \subset \mathbb{R}^d$ be a domain, let $\Sigma \subsetneq \partial\mathcal{O}$, and require that \mathcal{O} satisfy an interior sphere condition at each point of the boundary portion $\partial\mathcal{O} \setminus \Sigma$. Given functions $f \in C(\mathcal{O})$ and $h \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$, we call $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ a *solution* to a boundary value problem for a linear, second-order partial differential operator A in (1.3) with Neumann boundary condition along $\partial\mathcal{O} \setminus \Sigma$, if u obeys (2.4), (2.5), (2.6), and the derivative, $D_n u$, of u on $\partial\mathcal{O} \setminus \Sigma$ in the direction of the inward-pointing unit normal vector field, \vec{n} , is defined everywhere on $\partial\mathcal{O} \setminus \Sigma$, and

$$Au = f \quad \text{on } \mathcal{O}, \tag{4.10}$$

$$D_n u = h \quad \text{on } \partial\mathcal{O} \setminus \Sigma. \tag{4.11}$$

Corollary 4.12 (Uniqueness for the Neumann problem for C^2 functions on bounded domains). *Let $\mathcal{O} \subset \mathbb{R}^d$, and A , and $f \in C(\mathcal{O})$, and $h \in C_{\text{loc}}(\partial\mathcal{O} \setminus \Sigma)$ be as in Definition 4.11. Require in addition that \mathcal{O} be bounded and that $c > 0$ at some point of $\bar{\mathcal{O}} \setminus \Sigma$. If u_1, u_2 are solutions in the sense of Definition 4.11, then $u_1 = u_2$ on $\bar{\mathcal{O}}$.*

Proof. By hypothesis, $A(u_1 - u_2) = 0$ on \mathcal{O} and $D_n(u_1 - u_2) = 0$ on $\partial\mathcal{O} \setminus \Sigma$, so $u_1 - u_2 = 0$ on $\bar{\mathcal{O}}$ by Theorem 4.9. \square

5. WEAK MAXIMUM PRINCIPLE FOR SMOOTH FUNCTIONS

Having considered applications of the weak maximum principle property (Definition 2.8) to Dirichlet boundary value problems in §2, obstacle problems in §3 and, in conjunction with a Hopf Lemma for operators with non-negative characteristic form, to the strong maximum principle and Neumann boundary value problems in §4, we now establish conditions under which the operator A in (1.3) has the weak maximum principle property on $\mathcal{O} \cup \Sigma$. In §5.1, we establish a weak maximum principle for bounded C^2 functions on bounded domains (Theorem 5.1), while in §5.2, we extend this result to the case of bounded C^2 functions on unbounded domains (Theorem 5.3).

Our weak maximum principle (Theorems 5.1 and 5.3) differs in several aspects from [63, Theorem 1.1.2], some of which may appear subtle at first glance but which are nonetheless important for applications. For example,

- (1) The function u is *not* required to be in $C^2(\mathcal{O} \cup \Sigma) \cap C(\bar{\mathcal{O}})$, but rather $C^2(\mathcal{O}) \cap C(\bar{\mathcal{O}})$ and obey (2.4), (2.5), (2.6);
- (2) The subdomain $\mathcal{O} \subset \mathbb{R}^d$ is allowed to be *unbounded*; and
- (3) The coefficient functions of the partial differential operator A in (1.3) are allowed to be *unbounded*.

The significance of these points is illustrated further by the example of the Heston operator discussed in Appendix C.

5.1. Bounded C^2 functions on bounded domains. We begin with the case of bounded domains and adapt the proofs of [40, Theorem 3.1] and [48, Theorem 2.9.1]; see also [19, Theorem I.3.1], [30, §3.2]. It will be convenient to adopt the following convention. If $\Sigma \subseteq \partial\mathcal{O}$ and $g : \partial\mathcal{O} \setminus \Sigma \rightarrow \mathbb{R}$ is a function and $m \in \mathbb{R}$, then

$$m \vee \sup_{\partial\mathcal{O} \setminus \Sigma} g = \begin{cases} \sup_{\partial\mathcal{O} \setminus \Sigma} g & \text{if } \Sigma \subsetneq \partial\mathcal{O}, \\ m & \text{if } \Sigma = \partial\mathcal{O}, \end{cases} \quad (5.1)$$

where we recall that $x \vee y = \max\{x, y\}$, for any $x, y \in \mathbb{R}$.

Theorem 5.1 (Weak maximum principle for C^2 functions on bounded domains). *Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain, with Σ as in Definition 2.1. Require that the coefficients of the operator A in (1.3) be defined everywhere on $\bar{\mathcal{O}}$, obey (2.1) and¹¹*

$$\langle b, \vec{n} \rangle \geq 0 \quad \text{on } \Sigma, \quad (5.2)$$

$$c \geq 0 \quad \text{on } \mathcal{O} \cup \Sigma, \quad (5.3)$$

where Σ is assumed¹² to be C^1 with inward-pointing unit normal vector field \vec{n} . Assume further that at least one of the following holds,

$$\begin{cases} c > 0 \text{ on } \mathcal{O} \cup \Sigma, \text{ or} \\ \Sigma \neq \partial\mathcal{O} \text{ and } \langle b, h \rangle > 0 \text{ on } \Sigma \text{ and } \inf_{\mathcal{O}} \frac{\langle b, h \rangle}{\langle ah, h \rangle} > -\infty, \end{cases} \quad (5.4)$$

for some fixed direction, $h \in \mathbb{R}^d$, with $|h| = 1$.

¹¹The condition on c here is slightly stronger than that of (2.13).

¹²This assumption is required by the change-of-coordinates argument used in Step 1 of the proof.

Suppose that $u \in C^2(\mathcal{O}) \cap C(\bar{\mathcal{O}})$ obeys (2.4), (2.5), and (2.6). If u obeys (2.7), that is, $Au \leq 0$ on \mathcal{O} , then

$$\sup_{\mathcal{O}} u \leq 0 \vee \sup_{\partial\mathcal{O} \setminus \Sigma} u, \quad (5.5)$$

and, if $c = 0$ on $\mathcal{O} \cup \Sigma$ (and so $\Sigma \neq \partial\mathcal{O}$ by our hypothesis (5.4)), then

$$\sup_{\mathcal{O}} u = \sup_{\partial\mathcal{O} \setminus \Sigma} u. \quad (5.6)$$

Moreover, A has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ in the sense of Definition 2.8.

Remark 5.2 (Application to the elliptic Heston operator). The hypotheses of Theorem 5.1 on the coefficients of A are obeyed in the case of the elliptic Heston operator, Example 1.2, where $d = 2$ and $\mathcal{O} \subset \mathbb{H}$ is bounded and $\Sigma = \bar{\mathcal{O}} \cap \partial\mathbb{H}$. Choosing $h = e_2$ in condition (5.4), we have $b^2(x) = \kappa(\theta - x_2)$ and $a^{22}(x) = \sigma^2 x_2/2$, so

$$\frac{\langle b, e_2 \rangle}{\langle a e_2, e_2 \rangle} = \frac{b^2(x)}{a^{22}(x)} = \frac{2\kappa(\theta - x_2)}{\sigma^2 x_2} = \frac{2\kappa\theta}{\sigma^2 x_2} - \frac{2\kappa}{\sigma^2} > -\frac{2\kappa}{\sigma^2}, \quad \forall x_2 > 0,$$

and condition (5.2) is obeyed (with $\vec{n} = h$) as $\langle b, h \rangle = b^2(x_1, 0) = \kappa\theta > 0$. Lastly, $c = r \geq 0$. \square

Proof of Theorem 5.1. Since u is continuous on $\bar{\mathcal{O}}$ and by hypothesis $\bar{\mathcal{O}}$ is compact, then u achieves its maximum value at some point p in $\bar{\mathcal{O}}$.

Step 1 ($c > 0$ on $\mathcal{O} \cup \Sigma$). Suppose that u attains its maximum value, $u(p)$, at a point p in $\mathcal{O} \cup \Sigma$. If $p \in \mathcal{O}$, then $Du(p) = 0$ and $D^2u(p) \leq 0$, so that (2.1) gives

$$\begin{aligned} Au(p) &= -a^{ij}(p)u_{x_i x_j}(p) - b^i(p)u_{x_i}(p) + c(p)u(p) \\ &\geq c(p)u(p), \end{aligned}$$

since $a(p)$ is non-negative definite and $\text{tr}(KL) \geq 0$ whenever K, L are non-negative definite matrices [50, p. 218]. If $u(p) > 0$, we would obtain $Au(p) > 0$, contradicting our assumption (2.7) that $Au \leq 0$ on \mathcal{O} . Therefore, we must have $u(p) \leq 0$ and, necessarily, $u(p) \leq \sup_{\partial\mathcal{O} \setminus \Sigma} u^+$ when $\partial\mathcal{O} \setminus \Sigma$ is non-empty.

If $p \in \Sigma$ then, possibly after a C^2 change of coordinates on \mathbb{R}^d , we may assume without loss of generality that $B(p) \cap (\mathcal{O} \cup \Sigma) = B(p) \cap \mathbb{H}$, where $\mathbb{H} = \mathbb{R}^{d-1} \times \mathbb{R}_+$. Thus, $\vec{n}(p) = e_d$ and the condition (5.2) at p becomes $b^d(p) \geq 0$. We have $\text{tr}(aD^2u(p)) = 0$ by (2.6), while $u_{x_i}(p) = 0$ for $1 \leq i \leq d-1$ and $u_{x_d}(p) \leq 0$ since $p \in \Sigma$ is a local maximum and (2.4) holds, so that

$$\begin{aligned} Au(p) &= -a^{ij}(p)u_{x_i x_j}(p) - b^i(p)u_{x_i}(p) + c(p)u(p) \\ &= -b^d(p)u_{x_d}(p) + c(p)u(p) \\ &\geq c(p)u(p). \end{aligned}$$

Our assumption (2.7) that $Au \leq 0$ on \mathcal{O} implies, by continuity of Au on $\mathcal{O} \cup \Sigma$ via (2.4) and (2.5), that $Au \leq 0$ on $\mathcal{O} \cup \Sigma$. If $u(p) > 0$, we would obtain $Au(p) > 0$, a contradiction. Therefore, we must again have $u(p) \leq 0$ and, necessarily, $u(p) \leq \sup_{\partial\mathcal{O} \setminus \Sigma} u^+$ when $\partial\mathcal{O} \setminus \Sigma$ is non-empty.

Finally, if $p \in \partial\mathcal{O} \setminus \Sigma$, then $u(p) = \sup_{\partial\mathcal{O} \setminus \Sigma} u$ and so by combining the preceding three cases we obtain (5.5) for this step.

Step 2 ($c = 0$ on $\mathcal{O} \cup \Sigma$ and¹³ $\Sigma \neq \partial\mathcal{O}$ and $Au < 0$ on $\mathcal{O} \cup \Sigma$). Suppose that u attains its maximum value, $u(p)$, at a point p in $\mathcal{O} \cup \Sigma$. Repeating the argument of Step 1 would then yield

$$Au(p) \geq c(p)u(p) = 0,$$

contradicting the assumption that $Au < 0$ on \mathcal{O} . Consequently, we must have $p \in \partial\mathcal{O} \setminus \Sigma$ and

$$\sup_{\mathcal{O}} u = \sup_{\partial\mathcal{O} \setminus \Sigma} u,$$

that is, (5.6) holds.

Step 3 ($c = 0$ on $\mathcal{O} \cup \Sigma$ and $\Sigma \neq \partial\mathcal{O}$). We see that, for a constant $\nu > 0$ to be chosen and any $x \in \mathcal{O}$,

$$\begin{aligned} Ae^{\nu\langle h, x \rangle} &= (-\nu^2 a^{ij}(x) h_i h_j - \nu b^i(x) h_i) e^{\nu\langle h, x \rangle} \\ &= -\nu a^{ij}(x) h_i h_j \left(\nu + \frac{b^i(x) h_i}{a^{ij}(x) h_i h_j} \right) e^{\nu\langle h, x \rangle} \\ &< 0 \quad \text{on } \mathcal{O}, \quad \text{if } \inf_{x \in \mathcal{O}} \frac{b^i(x) h_i}{a^{ij}(x) h_i h_j} > -\nu, \end{aligned}$$

appealing to (5.4) for the ability to choose a finite $\nu > 0$. For any $x \in \Sigma$, we have

$$\begin{aligned} Ae^{\nu\langle h, x \rangle} &= -\nu b^i(x) h_i e^{\nu\langle h, x \rangle} \\ &< 0 \quad \text{on } \Sigma, \quad \text{if } b^i h_i > 0 \quad \text{on } \Sigma, \end{aligned}$$

again appealing to (5.4). Therefore, we have, for any $\varepsilon > 0$,

$$A(u + \varepsilon e^{\nu\langle h, x \rangle}) < 0 \quad \text{on } \mathcal{O} \cup \Sigma,$$

and so, by Step 2, we obtain

$$\sup_{\mathcal{O}} (u + \varepsilon e^{\nu\langle h, x \rangle}) = \sup_{\partial\mathcal{O} \setminus \Sigma} (u + \varepsilon e^{\nu\langle h, x \rangle}).$$

Letting $\varepsilon \rightarrow 0$ yields (5.6) for this step.

Step 4 ($c \geq 0$ on $\mathcal{O} \cup \Sigma$ and $\Sigma \neq \partial\mathcal{O}$). Let \mathcal{O}^+ denote the open subset $\{p \in \mathcal{O} : u(p) > 0\} \subset \mathcal{O}$. If \mathcal{O}^+ is empty, then $u \leq 0$ on \mathcal{O} and so $u \leq 0$ on $\bar{\mathcal{O}}$, with

$$\sup_{\mathcal{O}} u \leq 0 = \sup_{\partial\mathcal{O} \setminus \Sigma} u^+.$$

It remains to consider the case where \mathcal{O}^+ is non-empty. By hypothesis (2.7), $Au \leq 0$ on \mathcal{O} and thus $A_0 u \leq -cu \leq 0$ on \mathcal{O}^+ , where $A_0 := A - c$. We may write

$$\partial\mathcal{O}^+ = (\Sigma \cap \partial\mathcal{O}^+) \cup (\partial\mathcal{O}^+ \setminus \Sigma).$$

If $\partial\mathcal{O}^+ \setminus \Sigma$ were empty, then we would have $\partial\mathcal{O}^+ \subset \Sigma \subset \partial\mathcal{O}$. Thus, we would necessarily have $\mathcal{O}^+ = \mathcal{O}$ and hence $\partial\mathcal{O}^+ = \Sigma = \partial\mathcal{O}$, contradicting our assumption that $\Sigma \neq \partial\mathcal{O}$. Therefore, $\partial\mathcal{O}^+ \setminus \Sigma$ must be non-empty and our result (5.6) for the case $c = 0$ on $\mathcal{O} \cup \Sigma$ yields

$$\sup_{\mathcal{O}^+} u = \sup_{\partial\mathcal{O}^+ \setminus \Sigma} u.$$

¹³If $\Sigma = \partial\mathcal{O}$, then the conditions $Au < 0$ on Σ and $\text{tr}(aD^2u) = 0$ on Σ via (2.6) imply that $-\langle b, Du \rangle < 0$ on Σ . If in addition we had $b = \langle b, \vec{n} \rangle \vec{n}$ and (5.2) were strengthened to $\langle b, \vec{n} \rangle > 0$ on Σ , then we would obtain $D_n u > 0$ on Σ and consequently u would have a local maximum in \mathcal{O} , contradicting the condition $Au < 0$ on \mathcal{O} . In general, when $\Sigma = \partial\mathcal{O}$, one obtains no additional information regarding $\sup_{\mathcal{O}} u$ when $c = 0$ on $\mathcal{O} \cup \Sigma$, beyond the fact that u achieves its maximum at some point of $\mathcal{O} \cup \Sigma = \bar{\mathcal{O}}$.

Now

$$\partial\mathcal{O}^+ \setminus \Sigma = (\partial\mathcal{O}^+ \cap \partial\mathcal{O} \setminus \Sigma) \cup (\partial\mathcal{O}^+ \cap \mathcal{O}).$$

Since $u = 0$ on $\partial\mathcal{O}^+ \cap \mathcal{O}$ (because u is continuous on \mathcal{O} and $u \leq 0$ on $\mathcal{O} \setminus \mathcal{O}^+$), then

$$\sup_{\partial\mathcal{O}^+ \setminus \Sigma} u = 0 \vee \sup_{\partial\mathcal{O}^+ \cap \partial\mathcal{O} \setminus \Sigma} u = \sup_{\partial\mathcal{O} \setminus \Sigma} u,$$

and so, combining the preceding inequalities and equations and noting that $\sup_{\mathcal{O}} u = \sup_{\mathcal{O}^+} u$, we obtain

$$\sup_{\mathcal{O}} u = \sup_{\partial\mathcal{O} \setminus \Sigma} u,$$

as desired for the case of non-empty \mathcal{O}^+ . Combining the preceding cases yields (5.5) for this step.

This completes the proof. \square

5.2. Bounded C^2 functions on unbounded domains. Next, we consider the case of bounded C^2 functions on *unbounded* domains. We have the following refinement of the maximum principle for bounded C^2 functions on unbounded domains and elliptic operators with non-negative characteristic form [48, Theorem 2.9.2 & Exercises 2.9.4, 2.9.5].

Theorem 5.3 (Weak maximum principle for bounded C^2 functions on unbounded domains). *Assume that the coefficients of A in (1.3) obey the hypotheses of Theorem 5.1, except that the condition (5.4) on (a, b) or c is replaced by the condition (2.14) on c for some positive constant, c_0 , and, in addition, we require that there is a positive constant, K , such that*

$$\operatorname{tr} a(x) + \langle b(x), x \rangle \leq K(1 + |x|^2), \quad \forall x \in \mathcal{O}. \quad (5.7)$$

Suppose that $\mathcal{O} \subseteq \mathbb{R}^d$ is a possibly unbounded domain and that $u \in C^2(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}})$ obeys (2.4), (2.5), and (2.6). If $\sup_{\mathcal{O}} u < \infty$ and $u \leq 0$ on $\partial\mathcal{O} \setminus \Sigma$ (when non-empty), then

$$\sup_{\mathcal{O}} u \leq 0 \vee \frac{1}{c_0} \sup_{\mathcal{O}} Au.$$

Moreover, A has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ in the sense of Definition 2.8.

Proof. Let

$$v_0(x) := 1 + |x|^2, \quad \forall x \in \mathbb{R}^d, \quad (5.8)$$

and observe, noting that $c \geq 0$ on \mathcal{O} by hypothesis (2.14), that

$$\begin{aligned} Av_0(x) &= -2 \operatorname{tr} a(x) - 2 \langle b(x), x \rangle + c(x) (1 + |x|^2) \\ &\geq -2Kv_0(x), \quad \forall x \in \mathcal{O}, \end{aligned}$$

where K is the constant in (5.7). Therefore,

$$(A + 2K)v_0 \geq 0 \quad \text{on } \mathcal{O}. \quad (5.9)$$

Define

$$M := 0 \vee \sup_{\mathcal{O}} (A + 2K)u.$$

If $M = +\infty$, there is nothing to prove, so we may assume $0 \leq M < \infty$. For $\delta > 0$, set

$$w := u - \delta v_0 - (c_0 + 2K)^{-1} M, \quad (5.10)$$

and observe that, using $c \geq c_0 \geq 0$,

$$\begin{aligned} (A + 2K)w &= (A + 2K)u - \delta(A + 2K)v_0 - (c + 2K)(c_0 + 2K)^{-1}M \\ &\leq (A + 2K)u - M \\ &\leq 0 \quad \text{on } \mathcal{O}. \end{aligned}$$

Denoting $B(R) = \{x \in \mathbb{R}^d : |x| < R\}$, then for all large enough $R > 0$, we have $w \leq 0$ on $\mathcal{O} \cap \partial B(R)$ since u is bounded above on \mathcal{O} by hypothesis. Also, $u \leq 0$ on $\partial \mathcal{O} \setminus \Sigma$ (when non-empty) by hypothesis and so $w \leq 0$ on $\partial \mathcal{O} \setminus \Sigma$ (when non-empty). But

$$\partial(\mathcal{O} \cap B(R)) = (\bar{\mathcal{O}} \cap \partial B(R)) \cup (\bar{B}(R) \cap \partial \mathcal{O}),$$

and therefore,

$$\partial(\mathcal{O} \cap B(R)) \setminus \Sigma = (\bar{\mathcal{O}} \cap \partial B(R)) \cup (\bar{B}(R) \cap \partial \mathcal{O} \setminus \Sigma),$$

so that

$$w \leq 0 \quad \text{on } \partial(\mathcal{O} \cap B(R)) \setminus \Sigma.$$

Consequently, Theorem 5.1 (with A replaced by $A + 2K$) implies that $w \leq 0$ on $\mathcal{O} \cap B(R)$, for any sufficiently large $R > 0$ and thus $w \leq 0$ on \mathcal{O} . By letting $\delta \rightarrow 0$, we obtain $u \leq (c_0 + 2K)^{-1}M$ on \mathcal{O} and so

$$\begin{aligned} \sup_{\mathcal{O}} u^+ &\leq \frac{M}{c_0 + 2K} \\ &= \frac{1}{c_0 + 2K} \sup_{\mathcal{O}} ((A + 2K)u)^+ \\ &\leq \frac{1}{c_0 + 2K} \sup_{\mathcal{O}} (Au)^+ + \frac{2K}{c_0 + 2K} \sup_{\mathcal{O}} u^+, \end{aligned}$$

and thus

$$\frac{c_0}{c_0 + 2K} \sup_{\mathcal{O}} u^+ \leq \frac{1}{c_0 + 2K} \sup_{\mathcal{O}} (Au)^+,$$

which, using $c_0 > 0$, completes the proof. \square

Remark 5.4 (Application to the elliptic Heston operator). The hypotheses (2.14) and (5.7) in Theorem 5.3 are obeyed in the case of the elliptic Heston operator (Example 1.2) with

$$a(x) = \frac{x_2}{2} \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}, \quad b(x) = (r - q - x_2/2 \quad \kappa(\theta - x_2)), \quad \text{and} \quad c(x) = r,$$

provided $r > 0$. \square

Part 2. Weak maximum principles for bilinear maps and operators on functions in Sobolev spaces and applications to variational equations and inequalities

In this part of our article (§6 and §7), we develop weak maximum principles for bilinear maps and operators on functions in Sobolev spaces and applications to variational equations and inequalities.

6. APPLICATIONS OF THE WEAK MAXIMUM PRINCIPLE PROPERTY TO VARIATIONAL EQUATIONS

Just as in the case of the weak maximum principle for linear, second-order, partial differential operators A in (1.3) with non-negative characteristic form acting on smooth functions, we shall encounter many different situations (bounded or unbounded domains $\mathcal{O} \subset \mathbb{R}^d$, bounded or unbounded functions u with prescribed growth, and so on) where the basic weak maximum principle holds for bilinear maps \mathfrak{a} on $H^1(\mathcal{O}, \mathfrak{w})$ or associated linear operators $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$. Again, we find it useful to isolate that key property and then derive the consequences which necessarily follow in an essentially formal manner. In this section, we consider applications to variational equations. After providing some technical preliminaries, we proceed to the main applications, including the comparison principle (Proposition 6.7) and a priori estimates (Proposition 6.10) for $H^1(\mathcal{O}, \mathfrak{w})$ supersolutions and solutions to variational equations, and the corresponding results (Proposition 6.15) for $H^2(\mathcal{O}, \mathfrak{w})$ supersolutions and solutions to the associated boundary value problems. In §2.3, we show that when a bilinear map \mathfrak{a} on $H^1(\mathcal{O}, \mathfrak{w})$ or operator A on $H^2(\mathcal{O}, \mathfrak{w})$ has a weak maximum principle property for subsolutions (on unbounded domains) which are bounded above, the property may extend to subsolutions which instead obey a growth condition (Theorem 6.16 and Corollary 6.17).

Definition 6.1 (Weight function). Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a domain. We call \mathfrak{w} a *weight* function if

$$\mathfrak{w} \in C(\mathcal{O}) \cap L^1(\mathcal{O}) \quad \text{and} \quad \mathfrak{w} > 0 \quad \text{on } \mathcal{O}. \quad (6.1)$$

Definition 6.2 (Weighted Sobolev spaces). Given weight functions $\mathfrak{w}_{k,i}$, for integers $0 \leq i \leq k$, we define Hilbert spaces with norms

$$\|u\|_{H^k(\mathcal{O}, \mathfrak{w})}^2 := \sum_{i=0}^k \int_{\mathcal{O}} |D^i u|^2 \mathfrak{w}_{k,i} dx, \quad k \geq 1,$$

as the completions of the vector space $C_0^\infty(\bar{\mathcal{O}})$ with respect to the preceding norms and denote $L^2(\mathcal{O}, \mathfrak{w}) := H^0(\mathcal{O}, \mathfrak{w})$. Given $\Sigma \subseteq \partial\mathcal{O}$, we define $H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ to be the closure of $C_0^\infty(\mathcal{O} \cup \Sigma)$ in $H^1(\mathcal{O}, \mathfrak{w})$. \square

When $\Sigma = \partial\mathcal{O}$ in Definition 6.2, then $\mathcal{O} \cup \Sigma = \bar{\mathcal{O}}$ and $C_0^\infty(\mathcal{O} \cup \Sigma) = C_0^\infty(\bar{\mathcal{O}})$ and $H_0^1(\bar{\mathcal{O}}, \mathfrak{w}) = H^1(\mathcal{O}, \mathfrak{w})$. By analogy with [40, §8.1], for $u \in H^1(\mathcal{O}, \mathfrak{w})$, we define

$$\operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} u := \inf \{l \in \mathbb{R} : u \leq l \text{ on } \partial\mathcal{O} \setminus \Sigma \text{ in the sense of } H^1(\mathcal{O}, \mathfrak{w})\}, \quad (6.2)$$

where we recall that $u \leq l$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$ if $(u - l)^+ \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$. If $g \in H^1(\mathcal{O}, \mathfrak{w})$ and $m \in \mathbb{R}$, then

$$m \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} g := \begin{cases} \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} g & \text{if } \Sigma \subsetneq \partial\mathcal{O}, \\ m & \text{if } \Sigma = \partial\mathcal{O}, \end{cases} \quad (6.3)$$

by analogy with our convention (5.1) for everywhere-defined functions.

Definition 6.3 (Weak maximum principle property for a bilinear map). Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a domain, let

$$\mathfrak{a} : H^1(\mathcal{O}, \mathfrak{w}) \times H^1(\mathcal{O}, \mathfrak{w}) \rightarrow \mathbb{R},$$

be a bilinear map, and let $\Sigma \subseteq \partial\mathcal{O}$ be a relatively open subset¹⁴. We say that \mathfrak{a} obeys the *weak maximum principle property on $\mathcal{O} \cup \Sigma$* if whenever $u \in H^1(\mathcal{O}, \mathfrak{w})$ obeys

$$\begin{cases} \mathfrak{a}(u, v) \leq 0, \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}) \text{ with } v \geq 0 \text{ a.e. on } \mathcal{O}, \\ u \leq 0 \text{ on } \partial\mathcal{O} \setminus \Sigma \text{ in the sense of } H^1(\mathcal{O}, \mathfrak{w}), \end{cases}$$

and, if \mathcal{O} is unbounded, $\text{ess sup}_{\mathcal{O}} u < \infty$, then

$$u \leq 0 \quad \text{a.e. on } \mathcal{O}.$$

Remark 6.4 (Auxiliary boundedness condition in the definition of the weak maximum principle property). Once \mathfrak{a} is known to have the weak maximum principle property on $\mathcal{O} \cup \Sigma$, the condition that u be essentially bounded when \mathcal{O} is unbounded (which could be replaced by a growth condition) plays no role in most applications of the weak maximum principle property. \square

Remark 6.5 (Positivity conditions required for a bilinear map to have the weak maximum principle property on $\mathcal{O} \cup \Sigma$). We do not explicitly require that $\mathfrak{a}(1, \cdot) \geq 0$ (note that $1 \in H^1(\mathcal{O}, \mathfrak{w})$ by (6.1)), but we impose this requirement when needed in addition to our assumption that \mathfrak{a} has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ (compare [40, Equation (8.8)]). In order to prove that \mathfrak{a} has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ when \mathcal{O} is unbounded, we shall need to assume in addition that \mathfrak{a} obey the positivity condition (6.9). \square

Examples of bilinear maps, \mathfrak{a} , with the weak maximum principle property on $\mathcal{O} \cup \Sigma$ are provided by Theorems 8.2, 8.8, 8.11, 8.15, 6.16, and (when $\Sigma = \emptyset$) [40, Theorem 8.1].

By analogy with [28, §5.9.1], we let

$$H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w}) := (H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}))' \quad (6.4)$$

denote the dual space and, as in [28, §5.9.1], observe that

$$H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}) \subset L^2(\mathcal{O}, \mathfrak{w}) \subset H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w}).$$

The proofs of [3, Theorem 3.8] or [28, Theorem 5.9.1] easily adapt to show that every $F \in H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$ has the form

$$F(v) = (f, v)_{L^2(\mathcal{O}, \mathfrak{w})} - (f^i, v_{x_i})_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \quad (6.5)$$

and we write $F = (f, f^1, \dots, f^d)$, where $f, f^i \in L^2(\mathcal{O}, \mathfrak{w})$, $1 \leq i \leq d$. We say that $F \leq 0$ when

$$F(v) \leq 0, \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), v \geq 0 \text{ a.e. on } \mathcal{O},$$

and, given $F_1, F_2 \in H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$, we say that $F_1 \leq F_2$ if $F_1 - F_2 \leq 0$.

Definition 6.6 (Variational solution, subsolution, and supersolution). Let $F = (f, f^1, \dots, f^d) \in H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$ and $g \in H^1(\mathcal{O}, \mathfrak{w})$. We define $u \in H^1(\mathcal{O}, \mathfrak{w})$ to be a *variational subsolution*,

$$\mathfrak{a}(u, \cdot) \leq F,$$

$$u \leq g \quad \text{on } \partial\mathcal{O} \setminus \Sigma \text{ in the sense of } H^1(\mathcal{O}, \mathfrak{w}),$$

if $\text{ess sup}_{\mathcal{O}} u < \infty$ when \mathcal{O} is unbounded and

$$\mathfrak{a}(u, v) \leq F(v), \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), v \geq 0 \text{ a.e. on } \mathcal{O}, \quad (6.6)$$

$$(u - g)^+ \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}). \quad (6.7)$$

¹⁴We shall assume that $\Sigma \subseteq \partial\mathcal{O}$ is relatively open for the sake of consistency with Part 1 of our article, although Σ could now be any non-empty subset, not necessarily relatively open. However, the assumption of relative openness involves no loss of generality.

We call $u \in H^1(\mathcal{O}, \mathfrak{w})$ a *variational supersolution* if $-u$ is a variational subsolution and call $u \in H^1(\mathcal{O}, \mathfrak{w})$ a *variational solution* if it is both a variational subsolution and supersolution. \square

The first application of the weak maximum principle property, as in the case of Proposition 2.16, is to settle the question of *uniqueness*.

Proposition 6.7 (Comparison principle for variational supersolutions and uniqueness for variational solutions). *Let $F \in H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$, and $g \in H^1(\mathcal{O}, \mathfrak{w})$, and \mathfrak{a} be a bilinear map on $H^1(\mathcal{O}, \mathfrak{w})$ obeying the weak maximum principle property on $\mathcal{O} \cup \Sigma$, for some $\Sigma \subseteq \partial\mathcal{O}$. Suppose that $u \in H^1(\mathcal{O}, \mathfrak{w})$ is a subsolution and $u_2 \in H^1(\mathcal{O}, \mathfrak{w})$ a supersolution in the sense of Definition 6.6. Then $u_2 \geq u_1$ a.e. on \mathcal{O} and if u_1, u_2 are variational solutions, then $u_2 = u_1$ a.e. on \mathcal{O} .*

Proof. Since u_1 is a subsolution and u_2 a supersolution, then $u_1 - u_2$ is a subsolution with $u_1 - u_2 = 0$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$ and thus

$$u_1 - u_2 \leq 0 \quad \text{a.e. on } \mathcal{O},$$

because \mathfrak{a} has weak maximum principle property on $\mathcal{O} \cup \Sigma$. When u_1, u_2 are both solutions, then we also obtain $u_2 - u_1 \leq 0$ a.e. on \mathcal{O} and $u_2 = u_1$ a.e. on \mathcal{O} . \square

We shall occasionally need the following analogue of Hypotheses 2.17 and 2.18, respectively.

Hypothesis 6.8 (Non-negative zeroth-order coefficient). If \mathfrak{a} is a bilinear map on $H^1(\mathcal{O}, \mathfrak{w})$, then

$$\mathfrak{a}(1, v) \geq 0, \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), v \geq 0 \text{ a.e. on } \mathcal{O}. \quad (6.8)$$

Hypothesis 6.9 (Uniform positive lower bound on zeroth-order coefficient). If \mathfrak{a} is a bilinear map on $H^1(\mathcal{O}, \mathfrak{w})$, then there is a constant $c_0 > 0$ such that

$$\mathfrak{a}(1, v) \geq (c_0, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), v \geq 0 \text{ a.e. on } \mathcal{O}. \quad (6.9)$$

We can now proceed to give the expected a priori estimates.

Proposition 6.10 (Weak maximum principle and a priori estimates for $H^1(\mathcal{O}, \mathfrak{w})$ functions). *Let $F = (f, f^1, \dots, f^d) \in H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$, and $g \in H^1(\mathcal{O}, \mathfrak{w})$, and \mathfrak{a} be a bilinear map on $H^1(\mathcal{O}, \mathfrak{w})$ obeying the weak maximum principle property on $\mathcal{O} \cup \Sigma$, for some $\Sigma \subseteq \partial\mathcal{O}$, and assume that (6.8) holds. Suppose that $u \in H^1(\mathcal{O}, \mathfrak{w})$.*

- (1) *If $F \leq 0$ and u is a variational subsolution for F and g (Definition 6.6), then*

$$u \leq 0 \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \mathcal{O}.$$

- (2) *If $F = (f, 0, \dots, 0)$ and f has arbitrary sign and u is a variational subsolution for F and g but, in addition, there is a constant $c_0 > 0$ such that \mathfrak{a} obeys (6.9), then*

$$u \leq 0 \vee \frac{1}{c_0} \operatorname{ess\,sup}_{\mathcal{O}} f \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \mathcal{O}.$$

- (3) *If $F \geq 0$ and u is a variational supersolution for F and g (Definition 6.6), then*

$$u \geq 0 \wedge \operatorname{ess\,inf}_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \mathcal{O}.$$

- (4) *If $F = (f, 0, \dots, 0)$ and f has arbitrary sign, u is a variational supersolution for F and g , and \mathfrak{a} obeys (6.9), then*

$$u \geq 0 \wedge \frac{1}{c_0} \operatorname{ess\,inf}_{\mathcal{O}} f \wedge \operatorname{ess\,inf}_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \mathcal{O}.$$

(5) If $F = 0$ and u is a variational solution for F and g (Definition 6.6), then

$$\|u\|_{L^\infty(\mathcal{O})} \leq \|g\|_{L^\infty(\partial\mathcal{O}\setminus\Sigma)}.$$

(6) If $F = (f, 0, \dots, 0)$ and f has arbitrary sign, u is a variational solution for F and g , and \mathfrak{a} obeys (6.9), then

$$\|u\|_{L^\infty(\mathcal{O})} \leq \frac{1}{c_0} \|f\|_{L^\infty(\mathcal{O})} \vee \|g\|_{L^\infty(\partial\mathcal{O}\setminus\Sigma)},$$

The terms $\text{ess sup}_{\partial\mathcal{O}\setminus\Sigma} g$, and $\text{ess inf}_{\partial\mathcal{O}\setminus\Sigma} g$, and $\|g\|_{L^\infty(\partial\mathcal{O}\setminus\Sigma)}$ in the preceding items are omitted when $\Sigma = \partial\mathcal{O}$.

Proof. The proof follows almost the same pattern as that of Proposition 2.19. For Items (1) and (2), we describe the proof when $\Sigma \subsetneq \partial\mathcal{O}$; the proof for the case $\Sigma = \partial\mathcal{O}$ is the same except that an essential supremum of a non-negative function over $\partial\mathcal{O} \setminus \Sigma$ is replaced by zero. When f has arbitrary sign, choose

$$M := 0 \vee \frac{1}{c_0} \sup_{\mathcal{O}} f \vee \text{ess sup}_{\partial\mathcal{O}\setminus\Sigma} g,$$

while if $f \leq 0$ a.e. on \mathcal{O} , choose

$$M := 0 \vee \text{ess sup}_{\partial\mathcal{O}\setminus\Sigma} g.$$

We may assume without loss of generality that $M < \infty$. Since $M \geq g$ on $\partial\mathcal{O} \setminus \Sigma$ (in the sense of $H^1(\mathcal{O}, \mathfrak{w})$) and $c_0 M \geq f$ a.e. on \mathcal{O} , then, for all $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ with $v \geq 0$ a.e. on \mathcal{O} , the condition (6.9) gives

$$\mathfrak{a}(M, v) \geq (c_0 M, v)_{L^2(\mathcal{O}, \mathfrak{w})} \geq (f, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), v \geq 0 \text{ a.e. on } \mathcal{O},$$

when f has arbitrary sign and, when $f \leq 0$, the condition (6.8) gives

$$\mathfrak{a}(M, v) \geq 0 \geq (f, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), v \geq 0 \text{ a.e. on } \mathcal{O}.$$

Hence, M is a supersolution and thus $u \leq M$ a.e. on \mathcal{O} by Proposition 6.7, which gives Items (1) and (2).

Items (3) and (4) follow from Items (1) and (2) by noting that $-u$ is a subsolution for $-F$ and $-g$ if u is a supersolution for F and g . Item (5) follows by combining Items (1) and (3), while Item (6) follows by combining Items (2) and (4). \square

The a priori estimate in Item (6) may be compared with [63, Theorem 1.5.1 & 1.5.5] and [69, Lemma 2.8].

Definition 6.11 (Weak maximum principle property for a linear operator). Let $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ be a linear operator and let $\Sigma \subseteq \partial\mathcal{O}$. We say that A obeys the *weak maximum principle property* on $\mathcal{O} \cup \Sigma$ if whenever $u \in H^2(\mathcal{O}, \mathfrak{w})$

$$\begin{cases} Au \leq 0 \text{ a.e. on } \mathcal{O}, \\ u \leq 0 \text{ on } \partial\mathcal{O} \setminus \Sigma \text{ in the sense of } H^1(\mathcal{O}, \mathfrak{w}), \end{cases}$$

and, if \mathcal{O} is unbounded, $\text{ess sup}_{\mathcal{O}} u < \infty$, then

$$u \leq 0 \quad \text{a.e. on } \mathcal{O}.$$

Although the application we have in mind is where A is a linear, second-order partial differential operator whose principle symbol is zero on Σ , that situation is not assumed. The comments in Remarks 6.4 and 6.5 also apply to Definition 6.11.

Definition 6.12 (Integration by parts). Let $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ be a linear operator and let $\Sigma \subseteq \partial\mathcal{O}$. We say that a bilinear map $\mathfrak{a} : H^1(\mathcal{O}, \mathfrak{w}) \times H^1(\mathcal{O}, \mathfrak{w}) \rightarrow \mathbb{R}$ is associated to A on $\mathcal{O} \cup \Sigma$ through integration by parts if

$$(Au, v)_{L^2(\mathcal{O}, \mathfrak{w})} = \mathfrak{a}(u, v), \quad \forall u \in H^1(\mathcal{O}, \mathfrak{w}), v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}). \quad (6.10)$$

Examples of sufficient conditions for (6.10) to hold for bilinear maps, \mathfrak{a} , and operators, A , are described in §8.3. The following lemma, whose proof is clear, relates the weak maximum principle property on $\mathcal{O} \cup \Sigma$ in Definition 6.3 to that in Definition 6.11.

Lemma 6.13 (Relationship between weak maximum principle properties). *Let $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ be a linear operator, let $\Sigma \subseteq \partial\mathcal{O}$, and let $\mathfrak{a} : H^1(\mathcal{O}, \mathfrak{w}) \times H^1(\mathcal{O}, \mathfrak{w}) \rightarrow \mathbb{R}$ be a bilinear map which is associated to A on $\mathcal{O} \cup \Sigma$ through integration by parts. Then A obeys the weak maximum principle property on $\mathcal{O} \cup \Sigma$ if and only if the same is true for \mathfrak{a} .*

We have the following analogue of Definition 6.6.

Definition 6.14 (Strong solution, subsolution, and supersolution). Suppose $f \in L^2(\mathcal{O}, \mathfrak{w})$ and $g \in H^1(\mathcal{O}, \mathfrak{w})$. We define $u \in H^2(\mathcal{O}, \mathfrak{w})$ to be a *strong subsolution* if

$$Au \leq f \quad \text{a.e. on } \mathcal{O}, \quad (6.11)$$

$$u \leq g \quad \text{on } \partial\mathcal{O} \setminus \Sigma \text{ in the sense of } H^1(\mathcal{O}, \mathfrak{w}), \quad (6.12)$$

and $\text{ess sup}_{\mathcal{O}} u < \infty$ if \mathcal{O} is unbounded. We call $u \in H^2(\mathcal{O}, \mathfrak{w})$ a *strong supersolution* if $-u$ is a strong subsolution and a *strong solution* if it is both a strong subsolution and supersolution. \square

We then have

Proposition 6.15 (Weak maximum principle and a priori estimates for $H^2(\mathcal{O}, \mathfrak{w})$ functions). *Let $f \in L^2(\mathcal{O}, \mathfrak{w})$, and $g \in H^1(\mathcal{O}, \mathfrak{w})$, and $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ be a linear operator on $\mathcal{O} \cup \Sigma$ associated to a bilinear map \mathfrak{a} on $H^1(\mathcal{O}, \mathfrak{w})$ through integration by parts. Assume A obeys the weak maximum principle property on $\mathcal{O} \cup \Sigma$. Then the conclusions of Proposition 6.10 hold for functions $u \in H^2(\mathcal{O}, \mathfrak{w})$, provided the properties (6.8) or (6.9) for \mathfrak{a} are replaced by the properties for A that*

$$A1 \geq 0 \quad \text{a.e. on } \mathcal{O}, \quad (6.13)$$

or that there is a constant $c_0 > 0$ such that

$$A1 \geq c_0 \quad \text{a.e. on } \mathcal{O}, \quad (6.14)$$

and the role of Definition 6.6 is replaced by that of Definition 6.14.

Proof. When u is a strong subsolution (supersolution, solution), then it is necessarily a variational subsolution (supersolution, solution) using (6.10) and so the result follows immediately from Proposition 6.10. \square

Finally, we consider an application of the weak maximum principle property to unbounded functions on unbounded domains, by analogy with §2.3. Our Definition 6.3 of the weak maximum principle property requires that the subsolution, u , on an unbounded domain be essentially bounded above. However, if a bilinear map has the weak maximum principle property for subsolutions which are essentially bounded above, we may obtain an extension for subsolutions which instead obey a growth condition.

When $u \in C_0^\infty(\bar{\mathcal{O}})$ and $v \in C_0^\infty(\mathcal{O} \cup \Sigma)$, we have $\mathbf{a}(u, v) = (Au, v)_{L^2(\mathcal{O}, \mathfrak{w})}$. If $\varphi \in C^2(\mathcal{O})$ and $[A, \varphi]u = -B(\varphi u)$, as in (2.15), and

$$\begin{aligned} \mathbf{a}(\varphi u, v) &= (A\varphi u, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &= (\varphi Au, v)_{L^2(\mathcal{O}, \mathfrak{w})} + ([A, \varphi]u, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &= (Au, \varphi v)_{L^2(\mathcal{O}, \mathfrak{w})} - (B\varphi u, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &= \mathbf{a}(u, \varphi v)_{L^2(\mathcal{O}, \mathfrak{w})} - (B\varphi u, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \end{aligned}$$

and so

$$\mathbf{a}(\varphi u, v) + (B\varphi u, v)_{L^2(\mathcal{O}, \mathfrak{w})} = \mathbf{a}(u, \varphi v)_{L^2(\mathcal{O}, \mathfrak{w})}. \quad (6.15)$$

Let $B : H^1(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ be the first-order differential operator defined by (6.15) for all $u \in H^1(\mathcal{O}, \mathfrak{w})$ and $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$. We then have the following version of Theorem 2.20 for $H^1(\mathcal{O}, \mathfrak{w})$ functions.

Theorem 6.16 (Weak maximum principle for unbounded $H^1(\mathcal{O}, \mathfrak{w})$ functions on unbounded domains). *Let $\mathcal{O} \subseteq \mathbb{R}^d$ be an unbounded domain and let \mathbf{a} be a bilinear map on $H^1(\mathcal{O}, \mathfrak{w})$. Assume that the bilinear map,*

$$\hat{\mathbf{a}}(u, v) := \mathbf{a}(u, v) + (Bu, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall u, v \in H^1(\mathcal{O}, \mathfrak{w}), \quad (6.16)$$

obeys the weak maximum principle property on $\mathcal{O} \cup \Sigma$, for some $\Sigma \subseteq \partial\mathcal{O}$, for functions $u \in H^1(\mathcal{O}, \mathfrak{w})$ which are bounded above. Then, \mathbf{a} has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ for functions $u \in H^1(\mathcal{O}, \mathfrak{w})$ obeying the growth condition (2.17) a.e. on \mathcal{O} .

Proof. Since $\mathbf{a}(u, v) \leq 0$, the identities (6.15) and (6.16) imply that $\mathbf{a}(\varphi u, v) \leq 0$ for all $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$, $v \geq 0$ a.e. on \mathcal{O} . The conclusion now follows, just as in the proof of Theorem 2.20. \square

Corollary 6.17 (Weak maximum principle for unbounded $H^2(\mathcal{O}, \mathfrak{w})$ functions on unbounded domains). *Let $\mathcal{O} \subseteq \mathbb{R}^d$ be an unbounded domain and let $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ be a linear operator on $\mathcal{O} \cup \Sigma$, for some $\Sigma \subseteq \partial\mathcal{O}$, associated to a bilinear map \mathbf{a} on $H^1(\mathcal{O}, \mathfrak{w})$ through integration by parts. Assume that $\hat{A} : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ in (2.16) obeys the weak maximum principle property on $\mathcal{O} \cup \Sigma$ for functions $u \in H^2(\mathcal{O}, \mathfrak{w})$ which are bounded above. Then, A has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ for functions $u \in H^2(\mathcal{O}, \mathfrak{w})$ obeying the growth condition (2.17) a.e. on \mathcal{O} .*

Proof. When $u \in H^2(\mathcal{O}, \mathfrak{w})$ is a strong subsolution, then it is necessarily a variational subsolution since $\mathbf{a}(u, v) = (Au, v)_{L^2(\mathcal{O}, \mathfrak{w})}$ for all $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ by (8.31) and the result follows from Theorem 6.16. \square

7. APPLICATIONS OF THE WEAK MAXIMUM PRINCIPLE PROPERTY TO VARIATIONAL INEQUALITIES

We prove uniqueness for solutions to variational inequalities defined by bilinear maps \mathbf{a} on $H^1(\mathcal{O}, \mathfrak{w})$ and obstacle problems defined by bounded, linear operators A from $H^2(\mathcal{O}, \mathfrak{w})$ to $L^2(\mathcal{O}, \mathfrak{w})$, when \mathbf{a} or A obey the weak maximum principle property (Definition 6.3 or 6.11, respectively). Applications to variational inequalities are much simpler than in the case of the obstacle problems considered in §3 and include, in §7.1, a comparison principle for supersolutions and uniqueness for solutions to variational inequalities (Theorem 7.2) and, in §7.2, the corresponding results for $H^2(\mathcal{O}, \mathfrak{w})$ solutions to the obstacle problem (Theorem 7.7). In §7.3, we develop a

priori estimates (Proposition 7.9) for $H^1(\mathcal{O}, \mathfrak{w})$ supersolutions and solutions to variational inequalities and then the corresponding results for (Proposition 7.10) for $H^2(\mathcal{O}, \mathfrak{w})$ supersolutions and solutions to the obstacle problem.

7.1. Comparison principle for H^1 supersolutions and uniqueness for H^1 solutions to variational inequalities. We first recall our analogue [18] of the standard definition [9, 39, 46, 65, 69] of a solution and supersolution to a variational inequality.

Definition 7.1 (Solution and supersolution to a variational inequality). Given a source functional $F \in H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$, boundary data function $g \in H^1(\mathcal{O}, \mathfrak{w})$, and an obstacle function $\psi \in H^1(\mathcal{O}, \mathfrak{w})$ such that $\psi \leq g$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, that is,

$$(\psi - g)^+ \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \quad (7.1)$$

we say that a function $u \in H^1(\mathcal{O}, \mathfrak{w})$ is a *solution* to a variational inequality with Dirichlet boundary condition along $\partial\mathcal{O} \setminus \Sigma$ if $u = g$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, that is,

$$u - g \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}),$$

and

$$\begin{aligned} u &\geq \psi \text{ a.e. on } \mathcal{O} \quad \text{and} \quad \mathfrak{a}(u, v - u) \geq F(v - u), \\ \forall v &\in H^1(\mathcal{O}, \mathfrak{w}) \text{ with } v \geq \psi \text{ a.e. on } \mathcal{O} \text{ and } v - g \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \end{aligned} \quad (7.2)$$

and, if \mathcal{O} is unbounded, $\text{ess sup}_{\mathcal{O}} |u| < \infty$. We call a function $u \in H^1(\mathcal{O}, \mathfrak{w})$ a *supersolution* if $u \geq g$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, that is,

$$(u - g)^- \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}),$$

and

$$\begin{aligned} u &\geq \psi \text{ a.e. on } \mathcal{O} \quad \text{and} \quad \mathfrak{a}(u, w) \geq F(w), \\ \forall w &\in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}) \text{ with } w \geq 0 \text{ a.e. on } \mathcal{O}, \end{aligned} \quad (7.3)$$

and, if \mathcal{O} is unbounded, $\text{ess inf}_{\mathcal{O}} u > -\infty$.

If u is a solution in the sense of Definition 7.1 then we see that it is also a supersolution by writing $v = u + w$ and observing that $v \geq u \geq \psi$ a.e. on \mathcal{O} and $v = g$ on $\partial\mathcal{O} \setminus \Sigma$ if $w \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ and $w \geq 0$ a.e. on \mathcal{O} .

We now assume that the bilinear map \mathfrak{a} has the explicit form,

$$\mathfrak{a}(u, v) := \int_{\mathcal{O}} (a^{ij} u_{x_i} v_{x_j} + d^j u v_{x_j} - b^i u_{x_i} v + c u v) \mathfrak{w} \, dx, \quad \forall u, v \in C_0^\infty(\bar{\mathcal{O}}), \quad (7.4)$$

for some \mathfrak{w} as in Definition 6.1 (not necessarily coinciding with the \mathfrak{w}_i in the Definition 6.2 of $H^1(\mathcal{O}, \mathfrak{w})$) and where the coefficients a^{ij} , d^j , b^i , and c are Borel measurable functions on $\mathcal{O} \subset \mathbb{R}^d$. We require, in addition, that the coefficient matrix, $a = (a^{ij})$, obeys a version of (2.1) for Borel measurable functions, namely

$$\langle a\eta, \eta \rangle \geq 0 \quad \text{a.e. on } \mathcal{O}, \quad \forall \eta \in \mathbb{R}^d. \quad (7.5)$$

We then have the

Theorem 7.2 (Comparison principle for supersolutions and uniqueness for solutions to the variational inequality). *Let $F \in H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$, and $g \in H^1(\mathcal{O}, \mathfrak{w})$, and $\psi \in H^1(\mathcal{O}, \mathfrak{w})$, and \mathfrak{a} be a bilinear map of the form (7.4) on $H^1(\mathcal{O}, \mathfrak{w})$ obeying the weak maximum principle property on $\mathcal{O} \cup \Sigma$, for some $\Sigma \subseteq \partial\mathcal{O}$, and, in addition, that*

$$d^j = 0 \quad \text{on } \mathcal{O}, \quad 1 \leq j \leq d. \quad (7.6)$$

Suppose $u_1, u_2 \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ are a solution and supersolution, respectively, to the associated variational inequality (Definition 7.1). Then $u_2 \geq u_1$ a.e. on \mathcal{O} and if u_2 is also a solution, then $u_2 = u_1$ a.e. on \mathcal{O} .

We first consider the simpler case where the solution, supersolution, and obstacle function are continuous on \mathcal{O} , in which case we can adapt the proofs of [39, Theorems 1.3.3 & 1.3.4].

Proof of Theorem 7.2 for continuous functions. For this special case, the hypothesis (7.6) has no role and is not required. By replacing u by $u - g$, and v by $v - g$, and F by $F - \mathfrak{a}(g, \cdot)$, and ψ by $\psi - g$, if needed in Definition 7.1, we may assume without loss of generality that $g = 0$ on \mathcal{O} . We assume in addition to the hypotheses of Theorem 7.2 that $u_1, u_2, \psi \in C(\mathcal{O})$. Since u_2 is a supersolution, then

$$\mathfrak{a}(u_2, w) \geq F(w), \quad \forall w \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \quad w \geq 0 \text{ a.e. on } \mathcal{O},$$

while u_1 is a solution and thus obeys

$$\mathfrak{a}(u_1, v - u_1) \geq F(v - u_1), \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \quad v \geq \psi \text{ a.e. on } \mathcal{O}.$$

Choose $v = u_1 - w$, where $w \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ and $0 \leq w \leq u_1 - \psi$ a.e. on \mathcal{O} , so that $v \geq \psi$ a.e. on \mathcal{O} . If $u_1 = \psi$ a.e. on \mathcal{O} , then we can reverse the roles of u_1, u_2 and if $u_i = \psi$ a.e. on \mathcal{O} , $i = 1, 2$, we would be done, so we can assume without loss of generality that the open subset $G_i := \{P \in \mathcal{O} : u_i(P) > \psi(P)\} \subset \mathcal{O}$ is non-empty for $i = 1$ or 2 . We assume that $G_1 \neq \emptyset$ and note that

$$\mathfrak{a}(u_1, -w) \geq F(-w), \quad \forall w \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \quad 0 \leq w \leq u_1 - \psi \text{ a.e. on } \mathcal{O}.$$

Thus,

$$\mathfrak{a}(u_2 - u_1, w) \geq 0, \quad \forall w \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \quad 0 \leq w \leq u_1 - \psi \text{ a.e. on } \mathcal{O},$$

and in particular, noting that $G_1 = \{u_1 > \psi\}$ is non-empty and open and $u_1 - \psi \in C(G_1)$,

$$\mathfrak{a}(u_2 - u_1, w) \geq 0, \quad \forall w \in H_0^1(G_1 \cup \Sigma, \mathfrak{w}), \quad 0 \leq w \leq u_1 - \psi \text{ a.e. on } G_1.$$

By the weak maximum principle property, we obtain $u_2 \geq u_1$ on $G_1 \subset \mathcal{O}$ and thus $u_2 \geq u_1$ on \mathcal{O} , since $u_2 \geq \psi = u_1$ on $\mathcal{O} \setminus G_1$.

If u_1, u_2 are two solutions to (7.2), then the fact that u_2 is necessarily also a supersolution yields $u_2 \geq u_1$ a.e. on \mathcal{O} and reversing the roles of u_1, u_2 yields $u_1 \geq u_2$ a.e. on \mathcal{O} . This completes the proof of uniqueness for continuous solutions. \square

For the general case, we shall adapt the proof of [69, Theorem 4.27].

Proof of Theorem 7.2. As in the preceding case, we may assume without loss of generality that $g = 0$. Suppose that u_1 is a solution and u_2 is a supersolution and set $\hat{u} := (u_1 - u_2)^+$, so $u_1 \wedge u_2 = u_1 - (u_1 - u_2)^+ = u_1 - \hat{u}$. Observe that $\hat{u} \in H^1(\mathcal{O}, \mathfrak{w})$ and $u_1 - u_2 \leq 0$ on $\partial\mathcal{O} \setminus \Sigma$ and thus $\hat{u} = (u_1 - u_2)^+ = 0$ on $\partial\mathcal{O} \setminus \Sigma$, in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, and hence $\hat{u} \in H_0^1(\mathcal{O}, \mathfrak{w})$.

We claim that $\mathfrak{a}(\hat{u}, w) \leq 0$ for all $w \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ with $w \geq 0$ a.e. on \mathcal{O} . If not, there exists $w \in C_0^1(\mathcal{O} \cup \Sigma)$, $0 \leq w \leq 1$, such that

$$\mathfrak{a}(\hat{u}, w) > 0. \tag{7.7}$$

For $\varepsilon > 0$, define

$$w^\varepsilon := \frac{\hat{u}w}{\hat{u} + \varepsilon}.$$

Observe that $w^\varepsilon \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ and

$$w_{x_j}^\varepsilon = \frac{\hat{u}w_{x_j}}{\hat{u} + \varepsilon} + \frac{\varepsilon \hat{u}_{x_j} w}{(\hat{u} + \varepsilon)^2}, \quad 1 \leq j \leq d. \tag{7.8}$$

Choose $v^\varepsilon = u_1 - \varepsilon w^\varepsilon$ and observe that $v^\varepsilon \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ and

$$v^\varepsilon \geq u_1 - \hat{u}w \geq u_1 - \hat{u} = u_1 \wedge u_2 \geq \psi \quad \text{a.e. on } \mathcal{O}.$$

Because u_1 is a solution to the variational inequality (7.2), we have

$$\mathfrak{a}(u_1, v - u_1) \geq F(v - u_1), \quad \forall v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}) \text{ with } v \geq \psi \text{ a.e. on } \mathcal{O},$$

and thus, choosing $v = v^\varepsilon$ and using $v^\varepsilon - u_1 = -\varepsilon w^\varepsilon$, we obtain

$$\mathfrak{a}(-u_1, w^\varepsilon) \geq -F(w^\varepsilon).$$

But u_2 is a supersolution to the variational inequality (7.2), thus

$$\mathfrak{a}(u_2, w^\varepsilon) \geq F(w^\varepsilon).$$

Adding the preceding two inequalities yields

$$\mathfrak{a}(u_2 - u_1, w^\varepsilon) \geq 0.$$

Since $\hat{u} = (u_1 - u_2)^+$ and $w^\varepsilon = \hat{u}w/(\hat{u} + \varepsilon)$, we obtain

$$\mathfrak{a}(\hat{u}, w^\varepsilon) \leq 0.$$

Using (7.8), the expression (7.4) for \mathfrak{a} yields

$$\begin{aligned} \mathfrak{a}(\hat{u}, w^\varepsilon) &= \int_{\mathcal{O}} \left(a^{ij} \hat{u}_{x_i} w_{x_j}^\varepsilon + d^j \hat{u} w_{x_j}^\varepsilon - b^i \hat{u}_{x_i} w^\varepsilon + c \hat{u} w^\varepsilon \right) \mathfrak{w} dx \\ &= \int_{\mathcal{O}} \frac{\hat{u}}{\hat{u} + \varepsilon} \left(a^{ij} \hat{u}_{x_i} w_{x_j} + d^j \hat{u} w_{x_j} - b^i \hat{u}_{x_i} w + c \hat{u} w \right) \mathfrak{w} dx \\ &\quad + \varepsilon \int_{\mathcal{O}} \frac{w}{(\hat{u} + \varepsilon)^2} \left(a^{ij} \hat{u}_{x_i} + d^j \hat{u} \right) \hat{u}_{x_j} \mathfrak{w} dx \\ &=: I_1(\varepsilon) + \varepsilon I_2(\varepsilon). \end{aligned}$$

Consequently, by combining the preceding inequality and identity, we obtain

$$I_1(\varepsilon) + \varepsilon I_2(\varepsilon) \leq 0, \quad \forall \varepsilon > 0. \quad (7.9)$$

By hypothesis (7.6), we have $(d^j) = 0$ on \mathcal{O} and so the non-negative characteristic form condition (7.5) implies that

$$I_2(\varepsilon) = \int_{\mathcal{O}} \frac{w}{(\hat{u} + \varepsilon)^2} a^{ij} \hat{u}_{x_i} \hat{u}_{x_j} \mathfrak{w} dx \geq 0.$$

Therefore, $I_1(\varepsilon) \leq -\varepsilon I_2(\varepsilon) \leq 0$ for all $\varepsilon > 0$ and because

$$\lim_{\varepsilon \rightarrow 0} I_1(\varepsilon) = \mathfrak{a}(\hat{u}, w), \quad (7.10)$$

we obtain $\mathfrak{a}(\hat{u}, w) \leq 0$, contradicting our assumption that $\mathfrak{a}(\hat{u}, w) > 0$.

Thus, $\mathfrak{a}(\hat{u}, w) \leq 0$ for all $w \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ with $w \geq 0$ a.e. on \mathcal{O} and so the weak maximum principle property implies that $\hat{u} \leq 0$ a.e. on \mathcal{O} , that is $(u_1 - u_2)^+ = 0$ a.e. on \mathcal{O} and so $u_1 \leq u_2$ a.e. on \mathcal{O} . This completes the proof that if u_1 is a solution and u_2 a supersolution to the variational inequality (7.2), then $u_2 \geq u_1$ a.e. on \mathcal{O} .

If u_1, u_2 are two solutions to (7.2), then $u_1 = u_2$ a.e. on \mathcal{O} just as before. This completes the proof. \square

If we strengthen the condition (7.5), we can allow non-zero coefficients (d^j) in (7.4) for the statement and proof of Theorem 7.2. We constrain the behavior of the coefficients a, d in the expression (7.4) for the bilinear map \mathfrak{a} , near finite portions of $\partial\mathcal{O}$ as well as spatial infinity, with the aid of the

Definition 7.3 (Degeneracy coefficient). We call ϑ a *degeneracy coefficient* if

$$\vartheta \in C_{\text{loc}}(\bar{\mathcal{O}}) \quad \text{and} \quad \vartheta > 0 \quad \text{on } \mathcal{O}. \quad (7.11)$$

For our generalization of Theorem 7.2, we now require that the coefficients $(a^{ij}), (d^j)$ in (7.4) obey

$$\langle a\eta, \eta \rangle \geq \vartheta |\eta|^2 \quad \text{a.e. on } \mathcal{O}, \quad (7.12)$$

$$\langle a\eta, \eta \rangle \leq K\vartheta |\eta|^2 \quad \text{a.e. on } \mathcal{O}, \quad (7.13)$$

$$|\langle d, \eta \rangle| \leq K\vartheta |\eta| \quad \text{a.e. on } \mathcal{O}, \quad (7.14)$$

for all $\eta \in \mathbb{R}^2$. When the coefficients, (a^{ij}) , in (7.4) are only Borel measurable on $\bar{\mathcal{O}}$, our previous characterization (2.2) of the degeneracy locus is not meaningful and instead we have the

Definition 7.4 (Characterization of the degeneracy locus for a bilinear map). For a bilinear form as in (7.4), with measurable coefficients (a^{ij}) , the *degeneracy locus*, $\Sigma \subseteq \partial\mathcal{O}$, for \mathbf{a} is given by

$$\Sigma = \text{int}(\{x \in \partial\mathcal{O} : \vartheta(x) = 0\}). \quad (7.15)$$

Naturally, the definitions (2.2) and (7.15) coincide when $(a^{ij}) \in C_{\text{loc}}(\bar{\mathcal{O}}; \mathbb{R}^{d \times d})$.

Theorem 7.5 (Comparison principle for supersolutions and uniqueness for solutions to the variational inequality). *Assume the hypotheses of Theorem 7.2 but now require that either the coefficients (d^j) in the expression (7.4) for the bilinear map \mathbf{a} obey (7.6) or that the coefficients $(a^{ij}), (d^j)$ in (7.4) obey (7.12), (7.13), and (7.14) and that*

$$\vartheta \in L^1(\mathcal{O}; \mathfrak{w}). \quad (7.16)$$

Then the conclusions of Theorem 7.2 continue to hold.

Proof. We need only show that the assumption (7.7) still leads to a contradiction using the inequality (7.9) even when (d^j) is non-zero. By (7.7), (7.9), and (7.10), we obtain

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon I_2(\varepsilon) \leq -\lim_{\varepsilon \rightarrow 0} I_1(\varepsilon) = -\mathbf{a}(\hat{u}, w) < 0, \quad (7.17)$$

and so there is an $\varepsilon_0 > 0$ such that $\varepsilon I_2(\varepsilon) \leq 0$ and hence $I_2(\varepsilon) \leq 0$, $\forall \varepsilon \in (0, \varepsilon_0]$. Define

$$G_\varepsilon(\hat{u}) := \frac{w^{1/2} |D\hat{u}|}{\hat{u} + \varepsilon} \quad \text{a.e. on } \mathcal{O}.$$

Then, the fact that $I_2(\varepsilon) \leq 0$, $\forall \varepsilon \in (0, \varepsilon_0]$, yields

$$\begin{aligned} \int_{\mathcal{O}} \vartheta G_\varepsilon(\hat{u})^2 \mathfrak{w} \, dx &= \int_{\mathcal{O}} \vartheta \frac{w |D\hat{u}|^2}{(\hat{u} + \varepsilon)^2} \mathfrak{w} \, dx \leq \int_{\mathcal{O}} a^{ij} \frac{w \hat{u}_{x_i} \hat{u}_{x_j}}{(\hat{u} + \varepsilon)^2} \mathfrak{w} \, dx \quad (\text{by (7.12)}) \\ &\leq - \int_{\mathcal{O}} d^j \frac{w \hat{u}_{x_j}}{(\hat{u} + \varepsilon)^2} \mathfrak{w} \, dx \quad (\text{since } I_2(\varepsilon) \leq 0) \\ &\leq K \int_{\mathcal{O}} \vartheta \frac{w^{1/2} |D\hat{u}|}{\hat{u} + \varepsilon} \mathfrak{w} \, dx = K \int_{\mathcal{O}} \vartheta G_\varepsilon(\hat{u}) \mathfrak{w} \, dx \quad (\text{by (7.14)}) \\ &\leq K \left(\int_{\mathcal{O}} \vartheta \mathfrak{w} \, dx \right)^{1/2} \left(\int_{\mathcal{O}} \vartheta G_\varepsilon(\hat{u})^2 \mathfrak{w} \, dx \right)^{1/2} \\ &= C_0 \left(\int_{\mathcal{O}} \vartheta G_\varepsilon(\hat{u})^2 \mathfrak{w} \, dx \right)^{1/2}, \end{aligned}$$

where $C_0 := K \left(\int_{\mathcal{O}} \vartheta \mathfrak{w} dx \right)^{1/2}$. But then

$$\|\sqrt{\vartheta} G_{\varepsilon}(\hat{u})\|_{L^2(\mathcal{O}, \mathfrak{w})} \leq C_0, \quad \forall \varepsilon \in (0, \varepsilon_0],$$

and so, $\forall \varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned} |I_2(\varepsilon)| &\leq \int_{\mathcal{O}} \frac{w}{(\hat{u} + \varepsilon)^2} (|a^{ij} \hat{u}_{x_i} \hat{u}_{x_j}| + |d^j \hat{u}_{x_j}| \hat{u}) \mathfrak{w} dx \\ &\leq K \int_{\mathcal{O}} \vartheta \frac{w}{(\hat{u} + \varepsilon)^2} (|D\hat{u}|^2 + |D\hat{u}| \hat{u}) \mathfrak{w} dx \quad (\text{by (7.13) and (7.14)}) \\ &\leq K \int_{\mathcal{O}} \vartheta \frac{w |D\hat{u}|^2}{(\hat{u} + \varepsilon)^2} \mathfrak{w} dx + K \left(\int_{\mathcal{O}} \vartheta \frac{w |D\hat{u}|^2}{(\hat{u} + \varepsilon)^2} \mathfrak{w} dx \right)^{1/2} \left(\int_{\mathcal{O}} \vartheta \frac{w \hat{u}^2}{(\hat{u} + \varepsilon)^2} \mathfrak{w} dx \right)^{1/2} \\ &= K \|\sqrt{\vartheta} G_{\varepsilon}(\hat{u})\|_{L^2(\mathcal{O}, \mathfrak{w})}^2 + K \|\sqrt{\vartheta} G_{\varepsilon}(\hat{u})\|_{L^2(\mathcal{O}, \mathfrak{w})} \left(\int_{\mathcal{O}} \vartheta \mathfrak{w} dx \right)^{1/2} \\ &\leq (K + 1) C_0^2. \end{aligned}$$

Therefore, $|\varepsilon I_2(\varepsilon)| \leq (K + 1) C_0^2 \varepsilon$ and so $\lim_{\varepsilon \rightarrow 0} \varepsilon I_2(\varepsilon) = 0$, contradicting (7.17). \square

7.2. Comparison principle for H^2 supersolutions and uniqueness for H^2 solutions to obstacle problems. We recall our analogues of the standard definition [9, 39, 46, 65, 69] of a strong solution to an obstacle problem defined in [18].

Definition 7.6 (Strong solution and supersolution to an obstacle problem). Given functions $f \in L^2(\mathcal{O}, \mathfrak{w})$, $g \in H^1(\mathcal{O}, \mathfrak{w})$, and $\psi \in H^1(\mathcal{O}, \mathfrak{w})$ obeying (7.1), we call $u \in H^2(\mathcal{O}, \mathfrak{w})$ a *strong solution* to an obstacle problem for a linear operator, $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$, with Dirichlet boundary condition along $\partial\mathcal{O} \setminus \Sigma$, for some $\Sigma \subseteq \partial\mathcal{O}$, if

$$\min\{Au - f, u - \psi\} = 0 \quad \text{a.e. on } \mathcal{O}, \quad (7.18)$$

$$u - g \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \quad (7.19)$$

that is, $u = g$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, and, if \mathcal{O} is unbounded, $\text{ess sup}_{\mathcal{O}} |u| < \infty$. We call $u \in H^2(\mathcal{O}, \mathfrak{w})$ a *strong supersolution* if

$$\min\{Au - f, u - \psi\} \geq 0 \quad \text{a.e. on } \mathcal{O}, \quad (7.20)$$

$$(u - g)^- \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}), \quad (7.21)$$

that is, $u \geq g$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, and, if \mathcal{O} is unbounded, $\text{ess inf}_{\mathcal{O}} u > -\infty$.

Theorem 7.7 (Comparison principle and uniqueness for H^2 solutions to the obstacle problem). *Let $f \in L^2(\mathcal{O}, \mathfrak{w})$, and $g \in H^1(\mathcal{O}, \mathfrak{w})$, and $\psi \in H^1(\mathcal{O}, \mathfrak{w})$, and $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ be a linear operator on $\mathcal{O} \cup \Sigma$ associated to a bilinear map \mathfrak{a} of the form (7.4) on $H^1(\mathcal{O}, \mathfrak{w})$ through integration by parts, for some $\Sigma \subseteq \partial\mathcal{O}$. Assume A obeys the weak maximum principle property on $\mathcal{O} \cup \Sigma$ (Definition 6.11) and that the coefficients $(a^{ij}), (d^j)$ in the expression (7.4) obey the hypotheses of Theorem 7.5. Suppose $u_1 \in H^2(\mathcal{O}, \mathfrak{w})$ is a strong solution and $u_2 \in H^2(\mathcal{O}, \mathfrak{w})$ is a strong supersolution to the obstacle problem in the sense of Definition 7.6. Then $u_2 \geq u_1$ a.e. on \mathcal{O} and if u_1, u_2 are solutions, then $u_2 = u_1$ a.e. on \mathcal{O} .*

When the solution, $u_1, u_2 \in H^2(\mathcal{O}, \mathfrak{w})$, are also assumed to be continuous on \mathcal{O} , one can adapt the proof of [39, Theorem 1.3.4] to prove Theorem 7.7.

Proof of Theorem 7.7 for continuous solutions. We assume in addition to the hypotheses of Theorem 7.7 that $u_1, u_2 \in C(\mathcal{O})$. Let $\mathcal{U} = \{P \in \mathcal{O} : u_2(P) > u_1(P)\} \subset \mathcal{O}$. (If \mathcal{U} is empty, then $u_1 \leq u_2$ on \mathcal{O} and if u_1, u_2 are both solutions, we may reverse the roles of u_1 and u_2 to give $u_1 = u_2$ on \mathcal{O} .) Since $u_2 > u_1 \geq \psi$ on \mathcal{U} , then $Au_2 = f$ a.e. on \mathcal{U} and $Au_1 \geq f$ a.e. on \mathcal{O} implies that $A(u_2 - u_1) \leq 0$ a.e. on \mathcal{U} . Since $u_1 - u_2 = 0$ on $\mathcal{O} \cap \partial\mathcal{U}$, the weak maximum principle property implies that $u_2 \leq u_1$ on \mathcal{U} , a contradiction. \square

For the general case, we shall use the following analogue of the equivalence in [9, Equation (3.1.20)], whose proof is identical to the proof of the corresponding equivalence lemma in [18].

Lemma 7.8 (Equivalence of variational and strong (super-)solutions). *Assume the hypotheses for the operator, A , in Theorem 7.7. Let f , and g , and ψ be as in Definition 7.6 and suppose $u \in H^2(\mathcal{O}, \mathfrak{w})$. Then u is a (super-)solution to the variational inequality in Definition 7.1 if and only if u is a (super-)solution to the obstacle problem in Definition 7.6.*

Proof of Theorem 7.7. Since u_1, u_2 are a strong solution and supersolution in the sense of Definition 7.6, then they are necessarily a solution and supersolution to the variational inequality in the sense of Definition 7.1 by Lemma 7.8, and so $u_1 \leq u_2$ a.e. on \mathcal{O} by Theorem 7.5, while $u_1 = u_2$ a.e. on \mathcal{O} if both u_1 and u_2 are strong solutions. \square

7.3. A priori estimates for solutions and supersolutions to variational inequalities. We now state maximum principles and corresponding a priori estimates which extend those of non-coercive variational inequalities defined by uniformly elliptic partial differential operators, such as [65, Theorems 4.5.4 & 4.7.4] (see also [65, Theorem 4.5.1 & Corollary 4.5.2]). If $u \in H^1(\mathcal{O}, \mathfrak{w})$ is a supersolution to the variational inequality in Definition 7.1, then u necessarily obeys the inequality,

$$u \leq \psi \quad \text{a.e. on } \mathcal{O},$$

but the weak maximum principle yields additional a priori estimates for u , as described below.

Proposition 7.9 (A priori estimates for supersolutions and solutions to variational inequalities). *Let $F = (f, f^1, \dots, f^d) \in H^{-1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$, and $g \in H^1(\mathcal{O}, \mathfrak{w})$, and $\psi \in H^1(\mathcal{O}, \mathfrak{w})$, and \mathfrak{a} be a bilinear map of the form (7.4) on $H^1(\mathcal{O}, \mathfrak{w})$ obeying the hypotheses of Theorem 7.5, for some $\Sigma \subseteq \partial\mathcal{O}$, and assume that (6.8) holds. Suppose that $u \in H^1(\mathcal{O}, \mathfrak{w})$ is a supersolution to the associated variational inequality (Definition 7.1).*

(1) *If $F \geq 0$, then*

$$u \geq 0 \wedge \operatorname{ess\,inf}_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{a.e. on } \mathcal{O}.$$

(2) *If $F = (f, 0, \dots, 0)$ and f has arbitrary sign and \mathfrak{a} obeys (6.9), then*

$$u \geq 0 \wedge \frac{1}{c_0} \operatorname{ess\,inf}_{\mathcal{O}} f \wedge \operatorname{ess\,inf}_{\partial\mathcal{O} \setminus \Sigma} g \quad \text{on } \mathcal{O}.$$

(3) *If $F \leq 0$ and u is a solution for F and g and ψ (Definition 7.1), then*

$$u \leq 0 \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} g \vee \operatorname{ess\,sup}_{\mathcal{O}} \psi \quad \text{a.e. on } \mathcal{O}.$$

(4) *If $F = (f, 0, \dots, 0)$ and f has arbitrary sign, u is a solution for F and g and ψ , and \mathfrak{a} obeys (6.9), then*

$$u \leq 0 \vee \frac{1}{c_0} \operatorname{ess\,sup}_{\mathcal{O}} f \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} g \vee \operatorname{ess\,sup}_{\mathcal{O}} \psi \quad \text{on } \mathcal{O}.$$

- (5) If u_1 and u_2 are solutions, respectively, for $F_1 \geq F_2$ and $\psi_1 \geq \psi_2$ a.e. on \mathcal{O} , and $g_1 \geq g_2$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, then

$$u_1 \geq u_2 \quad \text{a.e. on } \mathcal{O}.$$

- (6) If u_1 and u_2 are solutions, respectively, for $F_1 \geq F_2$ and $\psi_1 \geq \psi_2$ a.e. on \mathcal{O} , and $g_1 \geq g_2$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, and \mathfrak{a} obeys (6.9), and $F_k = (f_k, 0, \dots, 0)$, $k = 1, 2$, then

$$\|u_1 - u_2\|_{L^\infty(\mathcal{O})} \leq \frac{1}{c_0} \|f_1 - f_2\|_{L^\infty(\mathcal{O})} \vee \|g_1 - g_2\|_{L^\infty(\partial\mathcal{O} \setminus \Sigma)} \vee \|\psi_1 - \psi_2\|_{L^\infty(\mathcal{O})},$$

while if $f_1 = f_2$ a.e. on \mathcal{O} and \mathfrak{a} obeys (6.8), then

$$\|u_1 - u_2\|_{L^\infty(\mathcal{O})} \leq \|g_1 - g_2\|_{L^\infty(\partial\mathcal{O} \setminus \Sigma)} \vee \|\psi_1 - \psi_2\|_{L^\infty(\mathcal{O})}.$$

The terms $\text{ess sup}_{\partial\mathcal{O} \setminus \Sigma} g$, and $\text{ess inf}_{\partial\mathcal{O} \setminus \Sigma} g$, and $\|g\|_{L^\infty(\partial\mathcal{O} \setminus \Sigma)}$, and $\|g_1 - g_2\|_{L^\infty(\partial\mathcal{O} \setminus \Sigma)}$ in the preceding items are omitted when $\Sigma = \partial\mathcal{O}$.

Proof. Consider Items (1) and (2). Since u is a supersolution to the variational inequality in Definition 7.1, then it is also a supersolution to the variational equation in Definition 6.6 (where ψ plays no role) and so Items (1) and (2) here just restate Items (3) and (4) in Proposition 6.10.

Consider Items (3) and (4). When $f \leq 0$ a.e. on \mathcal{O} , let

$$M := 0 \vee \text{ess sup}_{\partial\mathcal{O} \setminus \Sigma} g \vee \text{ess sup}_{\mathcal{O}} \psi,$$

while if f has arbitrary sign and \mathfrak{a} obeys (6.9), let

$$M := 0 \vee \frac{1}{c_0} \text{ess sup}_{\mathcal{O}} f \vee \text{ess sup}_{\partial\mathcal{O} \setminus \Sigma} g \vee \text{ess sup}_{\mathcal{O}} \psi.$$

We may assume without loss of generality that $M < \infty$. Then $M \geq \psi$ a.e. on \mathcal{O} and $M \geq g$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$, while for all $w \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ with $w \geq 0$ a.e. on \mathcal{O} , we have

$$\mathfrak{a}(M, w) = (cM, w)_{L^2(\mathcal{O}, \mathfrak{w})} \geq 0 \geq (f, w)_{L^2(\mathcal{O}, \mathfrak{w})},$$

when $f \leq 0$ a.e. on \mathcal{O} and

$$\mathfrak{a}(M, w) \geq (c_0 M, w)_{L^2(\mathcal{O}, \mathfrak{w})} \geq (f, w)_{L^2(\mathcal{O}, \mathfrak{w})},$$

when f has arbitrary sign. Hence, M is a supersolution and so Theorem 7.5 implies that $u \leq M$ a.e. on \mathcal{O} , which establishes Items (3) and (4).

The proofs of Items (5) and (6) here are very similar to the proofs of the corresponding Items (5) and (6) in Proposition 3.5, except that appeals to Proposition 3.3 are replaced by appeals to Theorem 7.5. \square

Note that an L^∞ comparison estimate for solutions u_1, u_2 corresponding to f, g , and obstacles ψ_1, ψ_2 is provided by [65, Theorem 4.7.4] and [9, Theorem 3.1.10].

Proposition 7.10 (Weak maximum principle and a priori estimates for strong solutions to obstacle problems). *Let $f \in L^2(\mathcal{O}, \mathfrak{w})$, and $g \in H^1(\mathcal{O}, \mathfrak{w})$, and $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ be a linear operator on $\mathcal{O} \cup \Sigma$ associated to a bilinear map \mathfrak{a} on $H^1(\mathcal{O}, \mathfrak{w})$ through integration by parts. Assume A obeys the weak maximum principle property on $\mathcal{O} \cup \Sigma$. Suppose that $u \in H^2(\mathcal{O}, \mathfrak{w})$. Then the conclusions of Proposition 7.9 hold, provided the properties (6.8) or (6.9) for \mathfrak{a} are replaced by the properties (6.13) or (6.14) for A , the role of Definition 7.1 is replaced by that of Definition 7.6, and supersolution is replaced by solution.*

Proof. When u is a strong (super-)solution, then it is necessarily a variational (super-)solution using (6.10) and so the result follows immediately from Proposition 7.9. \square

8. WEAK MAXIMUM PRINCIPLE FOR FUNCTIONS IN SOBOLEV SPACES

Having considered applications of the weak maximum principle property (Definition 6.3) to variational equations in §6 and variational inequalities in §7, we now establish conditions under which the bilinear map \mathfrak{a} on $H^1(\mathcal{O}, \mathfrak{w})$ or the associated differential operator $A : H^2(\mathcal{O}, \mathfrak{w}) \rightarrow L^2(\mathcal{O}, \mathfrak{w})$ has the weak maximum principle property on $\mathcal{O} \cup \Sigma$. We begin in §8.1 with some technical preliminaries. In §8.2, we prove a weak maximum principle for \mathfrak{a} when $H^1(\mathcal{O}, \mathfrak{w})$ is defined by power weights and \mathcal{O} is a bounded subdomain of the upper half-space, $\mathbb{H} \subset \mathbb{R}^d$ (Theorem 8.8); modulo a suitable weighted Sobolev inequality (Hypothesis 8.9), we then prove a more widely applicable weak maximum principle for \mathfrak{a} when $H^1(\mathcal{O}, \mathfrak{w})$ is defined by general weights and \mathcal{O} is a bounded subdomain of \mathbb{R}^d (Theorem 8.11). In §8.3, we describe an integration by parts formula relating, under suitable conditions on the weights and coefficients, the bilinear map, \mathfrak{a} , on $H^1(\mathcal{O}, \mathfrak{w})$ and the associated operator, A , on $H^2(\mathcal{O}, \mathfrak{w})$. Finally, in §8.4, we prove a weak maximum principle for bounded $H^1(\mathcal{O}, \mathfrak{w})$ functions on unbounded domains (Theorem 8.15).

Our weak maximum principle differs in several aspects from [63, Theorems 1.5.1 & 1.5.5], which again may appear subtle at first glance but which are still important for applications:

- (1) We use weighted Sobolev spaces adapted to the coefficients of the first and second-order derivatives in A and so those conditions are weaker than those of [63, Theorems 1.5.1 & 1.5.5] and permit applications to operators such as those of Examples 1.2 and 1.4 which are not covered by [63, Theorems 1.5.1 & 1.5.5];
- (2) The subdomain $\mathcal{O} \subset \mathbb{R}^d$ is allowed to be *unbounded*.

The differences between results obtainable from our maximum and comparison principles and those of Fichera, Oleřnik, and Radkevič are described in more detail in §C using the example of the Heston operator (Example 1.2).

Remark 8.1 (Strong maximum principle for bilinear maps on $H^1(\mathcal{O}, \mathfrak{w})$). We shall not consider in this article an analogue of the strong maximum principle for \mathfrak{a} on $H^1(\mathcal{O}, \mathfrak{w})$ corresponding to [40, Theorem 8.19], as such a result requires a corresponding version of the weak Harnack inequality [40, Theorem 8.18]. In the case of the Heston operator (Example 1.2) and its associated bilinear map, both a weak and a strong Harnack inequality (analogous to [40, Theorem 8.20]) are proved in [31] and we expect that those Harnack inequalities extend to the setting considered in this section.

8.1. Preliminaries. We let \mathfrak{w} be as in Definition 6.1 and ϑ be as in Definition 7.3 and obey (7.16) and choose

$$\|u\|_{H^1(\mathcal{O}, \mathfrak{w})}^2 := \int_{\mathcal{O}} (\vartheta |Du|^2 + (1 + \vartheta) |u|^2) \mathfrak{w} \, dx, \quad (8.1)$$

so that $\mathfrak{w}_{1,0} = (1 + \vartheta)\mathfrak{w}$ and $\mathfrak{w}_{1,1} = \vartheta\mathfrak{w}$ in the Definition 6.2 of $H^1(\mathcal{O}, \mathfrak{w})$, while $\mathfrak{w}_{0,0} = \mathfrak{w}$ in the definition of $L^2(\mathcal{O}, \mathfrak{w})$.

We shall assume that the coefficients, (a^{ij}) and (d^j) , of \mathfrak{a} in (7.4) obey (7.13) and (7.14) and, in addition, that the coefficients (b^i) obey

$$|\langle b, \eta \rangle| \leq K \vartheta |\eta| \quad \text{a.e. on } \mathcal{O}, \quad \forall \eta \in \mathbb{R}^d, \quad (8.2)$$

$$|c| \leq K(1 + \vartheta) \quad \text{a.e. on } \mathcal{O}, \quad (8.3)$$

for some positive constant, K . It is easy to check that \mathbf{a} obeys the *continuity estimate*,

$$\mathbf{a}(u, v) \leq C_1 \|u\|_{H^1(\mathcal{O}, \mathfrak{w})} \|v\|_{H^1(\mathcal{O}, \mathfrak{w})}, \quad \forall u, v \in H^1(\mathcal{O}, \mathfrak{w}), \quad (8.4)$$

for some positive constant, C_1 , when the coefficients, (a^{ij}) , (d^j) , and (b^i) , of \mathbf{a} in (7.4) obey (7.13), (7.14), (8.2), and (8.3), in which case $C_1 = C_1(K)$ in (8.4).

For $1 \leq p < \infty$, we let $L^p(\mathcal{O}, \mathfrak{w})$ denote the Banach space of Borel-measurable functions, u , on \mathcal{O} such that

$$\|u\|_{L^p(\mathcal{O}, \mathfrak{w})}^p := \int_{\mathcal{O}} |u|^p \mathfrak{w} \, dx < \infty. \quad (8.5)$$

The bilinear map \mathbf{a} obeys a *Gårding inequality*,

$$\mathbf{a}(u, u) \geq C_2 \|u\|_{H^1(\mathcal{O}, \mathfrak{w})}^2 - C_3 \|(1 + \vartheta)^{1/2} u\|_{L^2(\mathcal{O}, \mathfrak{w})}^2, \quad \forall u \in H^1(\mathcal{O}, \mathfrak{w}), \quad (8.6)$$

if, in addition, we require that the coefficient matrix, $a = (a^{ij})$, obeys (7.12), in which case $C_2 = C_2(K)$ and $C_3 = C_3(K)$.

We now specialize¹⁵ the Hilbert space $H^1(\mathcal{O}, \mathfrak{w})$ in Definition 6.2 to be the completion of the vector space $C_0^\infty(\bar{\mathcal{O}})$ with respect to the norm (8.1). Given $\Sigma \subseteq \partial\mathcal{O}$, we let $H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ be as in Definition 6.2.

8.2. H^1 functions on bounded and unbounded domains. We first consider a special case of our desired maximum principle, analogous to [40, Theorem 8.1].

Theorem 8.2 (Weak maximum principle for $H^1(\mathcal{O}, \mathfrak{w})$ functions). *Let $\mathcal{O} \subsetneq \mathbb{R}^d$ be a bounded domain and let $\Sigma \subseteq \partial\mathcal{O}$. Assume that the coefficients of \mathbf{a} in (7.4) are Borel-measurable on \mathcal{O} , obey (7.12), (7.13), (7.14), (6.8), (8.2), (8.3), and require that \mathfrak{w} and ϑ obey*

$$\inf_{\mathcal{O}} \vartheta \mathfrak{w} > 0 \text{ and } \sup_{\mathcal{O}} \vartheta \mathfrak{w} < \infty. \quad (8.7)$$

If $u \in H^1(\mathcal{O}, \mathfrak{w})$ obeys (6.6) with $f = 0$, then

$$\operatorname{ess\,sup}_{\mathcal{O}} u \leq 0 \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} u.$$

Moreover, \mathbf{a} has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ in the sense of Definition 6.3.

Remark 8.3 (Boundedness requirement on \mathcal{O}). In applications, the functions ϑ and \mathfrak{w} would not normally obey the upper bound in (8.7) unless \mathcal{O} were bounded and the lower bound in (8.7) unless $a = (a^{ij})$ were uniformly elliptic on \mathcal{O} . Moreover, boundedness of \mathcal{O} is implicitly used in the proof of Theorem 8.2 in inequalities involving the Lebesgue measures of the supports of functions and their gradients. However, unlike the proof of [40, Theorem 8.1], our proof of Theorem 8.2 avoids the use of the Poincaré inequality, thanks to a nice observation of Camelia Pop, and hence a requirement that \mathcal{O} is bounded originating from the usual statements of the Poincaré inequality (for example, [28, Theorem 5.6.3]).

Proof. We essentially follow the proof of [40, Theorem 8.1], but include the details here for later reference in our proof of Theorem 8.8. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ and $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$, then $uv \in W_0^{1,1}(\mathcal{O} \cup \Sigma, \mathfrak{w})$ and $D(uv) = vDu + uDv$ by analogy with [40, Problem 7.4], recalling that

¹⁵We prove an analogue of the Meyers-Serrin theorem [3] for unweighted Sobolev spaces in the context of certain weighted Sobolev spaces in [18].

$H^1(\mathcal{O}, \mathfrak{w}) = W_0^{1,2}(\mathcal{O}, \mathfrak{w})$ and $H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w}) = W_0^{1,2}(\mathcal{O} \cup \Sigma, \mathfrak{w})$. By the definition (7.4) of the bilinear map, $\mathfrak{a}(u, v)$, we obtain

$$\int_{\mathcal{O}} (a^{ij} u_{x_i} v_{x_j} - (d^i + b^i) u_{x_i} v) \mathfrak{w} dx \leq - \int_{\mathcal{O}} (d^i (uv)_{x_i} + cuv) \mathfrak{w} dx \leq 0,$$

for all $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ such that $v \geq 0$ and $uv \geq 0$ a.e. on \mathcal{O} , where to obtain the last inequality we use (6.8). Therefore,

$$\int_{\mathcal{O}} a^{ij} u_{x_i} v_{x_j} \mathfrak{w} dx \leq \int_{\mathcal{O}} (d^i + b^i) u_{x_i} v \mathfrak{w} dx \leq K \int_{\mathcal{O}} \vartheta |Du| v \mathfrak{w} dx \quad (\text{by (7.14) and (8.2)}).$$

Denote

$$l := 0 \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} u \geq 0,$$

and recall our convention (6.3) that $0 \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} u = 0$ when $\partial\mathcal{O} \setminus \Sigma = \emptyset$. We may assume without loss of generality that $l < \infty$, as otherwise there is nothing to prove. Suppose there exists a constant k such that

$$l \leq k < \operatorname{ess\,sup}_{\mathcal{O}} u \leq +\infty. \quad (8.8)$$

(If no such k exists, then we are done.) Set

$$v := (u - k)^+, \quad (8.9)$$

and observe that, because $u \leq k$ on $\partial\mathcal{O} \setminus \Sigma$ in the sense of $H^1(\mathcal{O}, \mathfrak{w})$ (when $\partial\mathcal{O} \setminus \Sigma$ is non-empty) and $u, k \in H^1(\mathcal{O}, \mathfrak{w})$, then $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ with

$$Dv = \begin{cases} Du & \text{for } u \geq k, \\ 0 & \text{for } u < k. \end{cases}$$

Consequently, if \mathcal{U} denotes the interior of $\operatorname{supp} Dv \subset \operatorname{supp} v$,

$$\int_{\mathcal{U}} \vartheta |Dv|^2 \mathfrak{w} dx \leq K \int_{\mathcal{U}} \vartheta |Dv| v \mathfrak{w} dx,$$

since $\langle Du, Dv \rangle = |Dv|^2$ a.e. on \mathcal{O} and (7.12) gives

$$\int_{\mathcal{O}} a^{ij} v_{x_i} v_{x_j} \mathfrak{w} dx \geq \int_{\mathcal{U}} \vartheta |Dv|^2 \mathfrak{w} dx.$$

Therefore, by the Cauchy-Schwartz inequality,

$$\|\vartheta^{1/2} Dv\|_{L^2(\mathcal{U}, \mathfrak{w})} \leq K \|\vartheta^{1/2} v\|_{L^2(\mathcal{U}, \mathfrak{w})}. \quad (8.10)$$

The inequality (8.10) yields

$$\|Dv\|_{L^2(\mathcal{U})} \leq C_1 \|v\|_{L^2(\mathcal{U})}, \quad (8.11)$$

for a positive constant

$$C_1 := K \left(\frac{\sup_{\mathcal{O}} \vartheta \mathfrak{w}}{\inf_{\mathcal{O}} \vartheta \mathfrak{w}} \right)^{1/2},$$

which is finite by (8.7). Combining the inequality (8.11) with the Sobolev embedding $W_0^{1,2}(\mathcal{U}) \rightarrow L^q(\mathcal{U})$, $2 \leq q < \infty$ if $d = 2$ and $2 \leq q \leq 2d/(d-2)$ if $d > 2$ [3, Theorem 5.4 (Parts I (A & B) &

III)] and the fact that $v \in W_0^{1,2}(\mathcal{U})$ implies, for positive constants C_2, C_3 depending on C_1, q, \mathcal{U} , that

$$\begin{aligned} \|v\|_{L^q(\mathcal{U})} &\leq C_2 (\|Dv\|_{L^2(\mathcal{U})} + \|v\|_{L^2(\mathcal{U})}) \quad (\text{Sobolev embedding with } q > 2) \\ &\leq C_3 \|v\|_{L^2(\mathcal{U})} \quad (\text{by (8.11)}) \\ &\leq C_3 |\mathcal{U}|^{1/2-1/q} \|v\|_{L^q(\mathcal{U})} \quad (\text{by [40, Equation (7.8)]}), \end{aligned}$$

and thus, recalling that \mathcal{U} is the interior of $\text{supp } Dv$,

$$|\text{supp } Dv|^{1/2-1/q} \geq C_3 > 0.$$

The inequality is independent of $k < \text{ess sup}_{\mathcal{O}} u$ and so continues to hold when we take the limit $k \rightarrow \text{ess sup}_{\mathcal{O}} u$. Hence, we see that u attains its maximum $\text{ess sup}_{\mathcal{O}} u \leq +\infty$ on a set, \mathcal{U} , of positive measure. If $\text{ess sup}_{\mathcal{O}} u = +\infty$ on \mathcal{U} , we obtain a contradiction to the fact that $u \in L^2(\mathcal{U})$, since $u \in L^2(\mathcal{O}, \mathfrak{w})$; if $\text{ess sup}_{\mathcal{O}} u < +\infty$ then, because u is constant on \mathcal{U} , one must also have $Du = 0$ on \mathcal{U} , contradicting the fact that

$$|\mathcal{U} \cap \text{supp } Du| = |\text{supp } Dv| > 0.$$

This completes the proof. \square

We now suppose $\mathcal{O} \subset \mathbb{H}$ and recall the following

Theorem 8.4 (Weighted Sobolev inequality for power weights). [47, Theorem 4.2.2] *Suppose $1 \leq p \leq q < \infty$, and $s > -1/p$, and $1 - d/p \leq \xi \leq 1$ is defined by*

$$\frac{1}{p} = \frac{1}{q} + \frac{1 - \xi}{d}. \quad (8.12)$$

Then there is a positive constant $C = C(d, p, q, s)$ such that, for any $u \in L^q(\mathbb{H}, x_d^s)$ with $Du \in L^p(\mathbb{H}, x_d^{s+\alpha}; \mathbb{R}^d)$, one has

$$\|x_d^s u\|_{L^q(\mathbb{H})} \leq C \|x_d^{s+\xi} Du\|_{L^p(\mathbb{H})}. \quad (8.13)$$

Remark 8.5. There is a minor typographical error in the statement of [47, Theorem 4.2.2].

We have the following generalization of [47, Lemma 4.2.4].

Corollary 8.6 (Application of weighted Sobolev inequality for power weights). *Suppose $\beta > 0$ and $2 - d \leq \alpha \leq 2$. For $u \in L^2(\mathbb{H}, x_d^{\beta-1})$ with $Du \in L^2(\mathbb{H}, x_d^{\beta-1+\alpha}; \mathbb{R}^d)$, and $q \geq 2$ defined by*

$$\frac{1}{2} = \frac{1}{q} + \frac{1 - \alpha/2}{d}, \quad (8.14)$$

and $2 \leq r \leq q$, one has

$$\|u\|_{L^r(\mathbb{H}, x_d^{\beta-1})} \leq C \|u\|_{L^2(\mathbb{H}, x_d^{\beta-1})}^\lambda \|Du\|_{L^2(\mathbb{H}, x_d^{\beta-1+\alpha})}^{1-\lambda}, \quad (8.15)$$

where $\lambda \in [0, 1]$ is defined by

$$\frac{1}{r} = \frac{\lambda}{2} + \frac{1 - \lambda}{q}. \quad (8.16)$$

Proof. When $s = (\beta - 1)/2$, so $\beta > 0$ when $s > -1/2$, and $\xi = \alpha/2$, so $1 - d/2 \leq \xi \leq 1$ when $2 - d \leq \alpha \leq 2$, it follows from Theorem 8.4 that

$$\|u\|_{L^q(\mathbb{H}, x_d^{\beta-1})} \leq C \|Du\|_{L^2(\mathbb{H}, x_d^{\beta-1+\alpha})}.$$

Holder's inequality, in the form of [40, Equation (7.9)], gives

$$\|u\|_{L^r(\mathbb{H}, x_d^{\beta-1})} \leq C \|u\|_{L^2(\mathbb{H}, x_d^{\beta-1})}^\lambda \|u\|_{L^q(\mathbb{H}, x_d^{\beta-1})}^{1-\lambda},$$

when $\lambda \in [0, 1]$ is defined by (8.16). Combining the preceding two inequalities yields the result. \square

Remark 8.7 (Weighted Poincaré and Sobolev inequalities). Direct weighted analogues of the standard Sobolev inequalities, with weights given by positive powers of the distance to a boundary portion, $\Sigma \subseteq \partial\mathcal{O}$, such as [49, Theorems 19.9 & 19.10], only hold for a very restrictive range of powers, even when \mathcal{O} is bounded. Koch provides a weighted Poincaré inequality on \mathbb{H} with a weight similar to ours [47, Lemma 4.4.4], as well as certain weighted Sobolev inequalities [47, Theorem 4.2.2 & Lemma 4.2.4]. Adams [3, §6.26] provides an unweighted Poincaré inequality which is valid on unbounded domains of “finite width”, while the Gagliardo-Nirenberg-Sobolev inequality [28, Theorem 5.6.1], a Poincaré-type inequality on \mathbb{R}^d and the Caffarelli-Kohn-Nirenberg inequality, another weighted Poincaré inequality on \mathbb{R}^d [21], are potentially useful in this context.

With the aid of Corollary 8.6, as noticed by Camelia Pop, we can relax the non-degeneracy requirement (8.7) in the hypotheses of Theorem 8.2.

Theorem 8.8 (Weak maximum principle for $H^1(\mathcal{O}, \mathfrak{w})$ functions on bounded subdomains of the upper half-space). *Assume the hypotheses of Theorem 8.2, except that we now omit the requirement (8.7). In addition, assume $\mathcal{O} \subsetneq \mathbb{H}$ is a bounded subdomain and require that there are constants, $0 < c_\vartheta < 1$ and $0 < c_{\mathfrak{w}} < 1$, such that*

$$c_\vartheta x_d^\alpha \leq \vartheta \leq c_\vartheta^{-1} x_d^\alpha \quad \text{on } \mathcal{O}, \quad (8.17)$$

$$c_{\mathfrak{w}} x_d^{\beta-1} \leq \mathfrak{w} \leq c_{\mathfrak{w}}^{-1} x_d^{\beta-1} \quad \text{on } \mathcal{O}, \quad (8.18)$$

where $\beta > 0$ and $0 \leq \alpha < 2$. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ obeys (6.6) with $f = 0$, then

$$\operatorname{ess\,sup}_{\mathcal{O}} u \leq 0 \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} u.$$

Moreover, \mathfrak{a} has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ in the sense of Definition 6.3.

Proof. We proceed as in the proof of Theorem 8.2, except that we now appeal to Corollary 8.6 in place of the standard Sobolev inequality [3, Theorem 5.4]. From (8.10) and the fact that v in (8.9) extends by zero outside $\operatorname{supp} v$ to an element of $H^1(\mathbb{H}, x_d^{\beta-1})$ by the analogue of [3, Lemma 3.22], also denoted by v , we obtain

$$\|Dv\|_{L^2(\mathcal{U}, x_d^{\beta-1+\alpha})} \leq C_1 \|v\|_{L^2(\mathcal{U}, x_d^{\beta-1+\alpha})}, \quad (8.19)$$

where $C_1 := (c_\vartheta c_{\mathfrak{w}})^{-1} K$ and, as in the proof of Theorem 8.2, the set $\mathcal{U} \subset \mathcal{O}$ denotes the interior of $\operatorname{supp} Dv$. Then

$$\begin{aligned} \|v\|_{L^r(\mathbb{H}, x_d^{\beta-1})} &\leq C_2 \|v\|_{L^2(\mathbb{H}, x_d^{\beta-1})}^\lambda \|Dv\|_{L^2(\mathbb{H}, x_d^{\beta-1+\alpha})}^{1-\lambda} \quad (\text{by (8.15)}) \\ &\leq C_3 \|v\|_{L^2(\mathbb{H}, x_d^{\beta-1})}^\lambda \|v\|_{L^2(\mathcal{U}, x_d^{\beta-1+\alpha})}^{1-\lambda} \quad (\text{by (8.19)}) \\ &\leq C_3 \max_{x \in \mathcal{O}} x_d^{(1-\lambda)\alpha/2} \|v\|_{L^2(\mathbb{H}, x_d^{\beta-1})}^\lambda \|v\|_{L^2(\mathcal{U}, x_d^{\beta-1})}^{1-\lambda}, \\ &\equiv C_4 \|v\|_{L^2(\mathbb{H}, x_d^{\beta-1})}^\lambda \|v\|_{L^2(\mathcal{U}, x_d^{\beta-1})}^{1-\lambda}, \end{aligned}$$

for a positive constant $C_4 \equiv C_3 \max_{x \in \mathcal{O}} x_d^{(1-\lambda)\alpha/2}$ independent of the constant k in (8.8), where we apply Corollary 8.6 with $r > 2$ (which is possible since $\alpha < 2$ and thus $q > 2$) and $0 < \lambda < 1$. Recall from [40, Equation (7.8)] that

$$\begin{aligned} \|v\|_{L^2(\mathbb{H}, x_d^{\beta-1})} &\leq |\text{supp } v|_\beta^{1/2-1/r} \|v\|_{L^r(\mathbb{H}, x_d^{\beta-1})}, \\ \|v\|_{L^2(\mathcal{U}, x_d^{\beta-1})} &\leq |\mathcal{U}|_\beta^{1/2-1/r} \|v\|_{L^r(\mathbb{H}, x_d^{\beta-1})}, \end{aligned}$$

where we denote $|\mathcal{S}|_\beta := \int_{\mathcal{S}} x_d^{\beta-1} dx$ for any $\beta > 0$ and Borel-measurable subset $\mathcal{S} \subset \mathbb{H}$. Hence, the preceding inequalities give

$$\begin{aligned} \|v\|_{L^r(\mathbb{H}, x_d^{\beta-1})} &\leq C_4 \|v\|_{L^2(\mathbb{H}, x_d^{\beta-1})}^\lambda \|v\|_{L^2(\mathcal{U}, x_d^{\beta-1})}^{1-\lambda} \\ &\leq C_4 |\text{supp } v|_\beta^{\lambda(1/2-1/r)} \|v\|_{L^r(\mathbb{H}, x_d^{\beta-1})}^\lambda |\mathcal{U}|_\beta^{(1-\lambda)(1/2-1/r)} \|v\|_{L^r(\mathcal{U}, x_d^{\beta-1})}^{1-\lambda} \\ &= C_4 |\text{supp } v|_\beta^{\lambda(1/2-1/r)} |\mathcal{U}|_\beta^{(1-\lambda)(1/2-1/r)} \|v\|_{L^r(\mathcal{U}, x_d^{\beta-1})}, \end{aligned}$$

and so, noting that \mathcal{U} is the interior of $\text{supp } Dv$,

$$C_4 |\text{supp } v|_\beta^{\lambda(1/2-1/r)} |\text{supp } Dv|_\beta^{(1-\lambda)(1/2-1/r)} \geq 1.$$

Thus, since $|\text{supp } v|_\beta \leq |\mathcal{O}|_\beta < \infty$ and $|\mathcal{O}|_\beta > 0$,

$$|\text{supp } Dv|_\beta^{(1-\lambda)(1/2-1/r)} \geq C_4^{-1} |\mathcal{O}|_\beta^{-\lambda(1/2-1/r)} > 0,$$

recalling that $0 < \lambda < 1$ and $r > 2$. We again obtain a contradiction, after taking the limit $k \rightarrow \text{ess sup}_{\mathcal{O}} u$, and the result follows. \square

Given a suitable weighted Sobolev inequality for functions on subdomains $\mathcal{O} \subseteq \mathbb{R}^d$ (see Corollary B.2 for a general class of examples for functions on \mathbb{R}^d), it is a straightforward to generalize Theorem 8.8 from the case where ϑ and \mathfrak{w} obey (8.17) and (8.18).

Hypothesis 8.9 (Weighted Sobolev inequality). Given a subdomain $\mathcal{O} \subseteq \mathbb{R}^d$, constants $1 \leq p \leq q < \infty$, and functions $\vartheta, \mathfrak{w} \in C(\mathcal{O})$ such that $\vartheta > 0$ and $\mathfrak{w} > 0$ on \mathcal{O} , there is a positive constant $C = C(p, q, \vartheta, \mathfrak{w})$ such that, for any $u \in L^q(\mathcal{O}, \mathfrak{w})$ with $Du \in L^p(\mathcal{O}, \vartheta \mathfrak{w}; \mathbb{R}^d)$, one has

$$\|u\|_{L^q(\mathcal{O}, \mathfrak{w})} \leq C \|Du\|_{L^p(\mathcal{O}, \vartheta \mathfrak{w})}. \quad (8.20)$$

Corollary 8.10 (Application of weighted Sobolev inequality). Assume Hypothesis 8.9 holds with $p = 2$. For $u \in L^2(\mathcal{O}, \mathfrak{w})$ with $Du \in L^2(\mathcal{O}, \vartheta \mathfrak{w}; \mathbb{R}^d)$ and $2 \leq r \leq q$, one has

$$\|u\|_{L^r(\mathcal{O}, \mathfrak{w})} \leq C \|u\|_{L^2(\mathcal{O}, \mathfrak{w})}^\lambda \|Du\|_{L^2(\mathcal{O}, \vartheta \mathfrak{w})}^{1-\lambda}, \quad (8.21)$$

where $\lambda \in [0, 1]$ is defined by (8.16).

Proof. The proof is similar to that of Corollary 8.6. Inequality (8.20), with $2 \leq q < \infty$, gives

$$\|u\|_{L^q(\mathcal{O}, \mathfrak{w})} \leq C \|Du\|_{L^2(\mathcal{O}, \vartheta \mathfrak{w})}.$$

Holder's inequality, in the form of [40, Equation (7.9)], yields

$$\|u\|_{L^r(\mathcal{O}, \mathfrak{w})} \leq C \|u\|_{L^2(\mathcal{O}, \mathfrak{w})}^\lambda \|u\|_{L^q(\mathcal{O}, \mathfrak{w})}^{1-\lambda},$$

when $\lambda \in [0, 1]$ is defined by (8.16). Combining the preceding two inequalities yields the result. \square

We have the following generalization of Theorem 8.2.

Theorem 8.11 (Weak maximum principle for $H^1(\mathcal{O}, \mathfrak{w})$ functions on bounded domains and general weights). *Assume the hypotheses of Theorem 8.2, except that we allow $\mathcal{O} \in \mathbb{R}^d$ to be any bounded subdomain with $\Sigma \subseteq \partial\mathcal{O}$ and allow ϑ, \mathfrak{w} to be any functions obeying Hypothesis 8.9 for $p = 2$ and some $2 < q < \infty$. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ obeys (6.6) with $f = 0$, then*

$$\operatorname{ess\,sup}_{\mathcal{O}} u \leq 0 \vee \operatorname{ess\,sup}_{\partial\mathcal{O} \setminus \Sigma} u.$$

Moreover, \mathfrak{a} has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ in the sense of Definition 6.3.

Proof. The proof is almost identical to that of Theorems 8.2 and 8.8, except for a few minor changes which we indicate here. In place of (8.19), we note that (8.10) may be written as

$$\|Dv\|_{L^2(\mathcal{U}, \vartheta \mathfrak{w})} \leq K \|v\|_{L^2(\mathcal{U}, \vartheta \mathfrak{w})}. \quad (8.22)$$

We now proceed as in the proof of Theorem 8.8, but apply (8.21) in place of (8.15) and choose $C_4 = C_3 \max_{x \in \mathcal{O}} \vartheta^{(1-\lambda)/2}$ instead of $C_3 \max_{x \in \mathcal{O}} x_d^{(1-\lambda)\alpha/2}$. \square

8.3. Integration by parts formula. Before proceeding to consider when the weak maximum principle holds on unbounded domains, we shall need to introduce an integration by parts formula and so, to accomplish this, we must impose additional conditions on the coefficients of \mathfrak{a} beyond those stated in §8.1. For now, we shall require that $a = (a^{ij})$ be continuous on $\bar{\mathcal{O}}$ (not merely Borel measurable on \mathcal{O}) but shortly strengthen this condition further. Let A be the partial differential operator given in equivalent divergence and non-divergence forms by

$$\begin{aligned} Au &:= - (a^{ij} u_{x_i} + d^j u)_{x_j} - (b^i + (\log \mathfrak{w})_{x_j} a^{ij}) u_{x_i} + cu \\ &= -a^{ij} u_{x_i x_j} - \left(b^i + a_{x_j}^{ij} + d^i + (\log \mathfrak{w})_{x_j} a^{ij} \right) u_{x_i} + \left(c - d_{x_j}^j - (\log \mathfrak{w})_{x_j} d^j \right) u \\ &= -a^{ij} u_{x_i x_j} - \tilde{b}^i u_{x_i} + \tilde{c} u, \end{aligned} \quad (8.23)$$

with

$$\tilde{b}^i := b^i + a_{x_j}^{ij} + (\log \mathfrak{w})_{x_j} a^{ij}, \quad 1 \leq i \leq d, \quad (8.24)$$

$$\tilde{c} := c - d_{x_j}^j - (\log \mathfrak{w})_{x_j} d^j, \quad (8.25)$$

and where we now impose the additional regularity requirements,

$$a^{ij} \in C^{0,1}(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}}), \quad 1 \leq i, j \leq d, \quad (8.26)$$

$$d^j \in C^{0,1}(\mathcal{O}) \cap C_{\text{loc}}(\bar{\mathcal{O}}), \quad 1 \leq j \leq d, \quad (8.27)$$

$$\log \mathfrak{w} \in C^{0,1}(\mathcal{O}). \quad (8.28)$$

Provided we also require that ϑ, \mathfrak{w} obey

$$\vartheta \mathfrak{w} \in C_{\text{loc}}(\bar{\mathcal{O}}), \quad (8.29)$$

$$\vartheta \mathfrak{w} = 0 \text{ on } \Sigma, \quad (8.30)$$

and that \mathcal{O} is a domain for which the divergence theorem holds, then integration by parts in (7.4) yields the integration by parts relation (6.10) when $u \in C_0^\infty(\bar{\mathcal{O}})$ and $v \in C_0^\infty(\mathcal{O} \cup \Sigma)$, that is,

$$\mathfrak{a}(u, v) = (Au, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad (8.31)$$

since

$$\begin{aligned}
\mathfrak{a}(u, v) &= \int_{\mathcal{O}} ((a^{ij}u_{x_i} + d^j u) v_{x_j} - b^i u_{x_i} v + cuv) \mathfrak{w} dx \quad (\text{by (7.4)}) \\
&= \int_{\mathcal{O}} \left(- (a^{ij}u_{x_i} + d^j u)_{x_j} - b^i u_{x_i} + cu \right) v \mathfrak{w} dx \\
&\quad - \int_{\mathcal{O}} (a^{ij}u_{x_i} + d^j u) v \mathfrak{w}_{x_j} dx - \int_{\partial \mathcal{O}} n_j (a^{ij}u_{x_i} + d^j u) v \mathfrak{w} ds \\
&= \int_{\mathcal{O}} \left(-a^{ij}u_{x_i x_j} - \left(b^i + a_{x_j}^{ij} + d^i + (\log \mathfrak{w})_{x_j} a^{ij} \right) u_{x_i} \right. \\
&\quad \left. + \left(c - d_{x_j}^j - (\log \mathfrak{w})_{x_j} d^j \right) u \right) v \mathfrak{w} dx \\
&= (Au, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad (\text{by (8.23)})
\end{aligned}$$

where \vec{n} is the *inward*-pointing normal vector field and the integral over $\partial \mathcal{O}$ is zero since $v = 0$ on $\partial \mathcal{O} \setminus \Sigma$ and \mathfrak{w}, ϑ obey (8.29) and (8.30), and the coefficients $(a^{ij}), (d^j)$ obey (7.12) and (7.13) on $\bar{\mathcal{O}}$ together with (8.26) and (8.27).

Remark 8.12 (Relaxing the conditions on $\partial \mathcal{O}$). More generally, if the divergence theorem is not assumed to hold for \mathcal{O} , then (8.31) still holds under slightly stronger regularity assumptions on the coefficients, a^{ij} , and weight, \mathfrak{w} , near Σ .

Clearly, when the coefficients $(a, \tilde{b}, \tilde{c})$ of A obey (7.13) and

$$|\tilde{b}| \leq K(1 + \vartheta) \quad \text{a.e. on } \mathcal{O}, \quad (8.32)$$

$$|\tilde{c}| \leq K(1 + \vartheta) \quad \text{a.e. on } \mathcal{O}, \quad (8.33)$$

there is a positive constant, $C_4 = C_4(K)$, such that

$$\|Au\|_{L^2(\mathcal{O}, \mathfrak{w})} \leq C_4 \|u\|_{H^2(\mathcal{O}, \mathfrak{w})}, \quad \forall u \in C_0^\infty(\bar{\mathcal{O}}), \quad (8.34)$$

where we set

$$\|u\|_{H^2(\mathcal{O}, \mathfrak{w})}^2 := \int_{\mathcal{O}} (\vartheta^2 |D^2 u|^2 + (1 + \vartheta^2) (|Du|^2 + |u|^2)) \mathfrak{w} dx, \quad (8.35)$$

so that $\mathfrak{w}_{2,0} = \mathfrak{w}_{2,1} = (1 + \vartheta)^2 \mathfrak{w}$ and $\mathfrak{w}_{2,2} = \vartheta^2 \mathfrak{w}$ in the Definition 6.2 of $H^2(\mathcal{O}, \mathfrak{w})$.

We specialize the Hilbert space $H^2(\mathcal{O}, \mathfrak{w})$ in Definition 6.2 to be the completion of the vector space $C_0^\infty(\bar{\mathcal{O}})$ with respect to the norm (8.35) and note that (8.34) continues to hold when $u \in H^2(\mathcal{O}, \mathfrak{w})$. Furthermore, the proof of [18, Lemma 2.33] (integration by parts) adapts to show that (8.31) continues to hold when $u \in H^2(\mathcal{O}, \mathfrak{w})$ and $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$.

Remark 8.13 (Conditions on the coefficients of A and the integral weight). The bounds (8.32) and (8.33) hold, for example, if we strengthen the conditions (8.26), (8.27), and (8.28) by requiring that $|a_{x_j}^{ij}| \leq K(1 + \vartheta)$ a.e. on \mathcal{O} , and $|d_{x_j}^j| \leq K(1 + \vartheta)$ a.e. on \mathcal{O} , and $|(\log \mathfrak{w})_{x_j}| \leq K(1 + \vartheta)$ a.e. on \mathcal{O} .

Example 8.14 (Heston operator, bilinear map, and integration by parts formula). We show in §C that the coefficients of the Heston operator, A , its associated bilinear map, \mathfrak{a} , and weight function, \mathfrak{w} , defined in [18], and the degeneracy coefficient, ϑ , obey the conditions described in §8.3.

8.4. H^1 functions on unbounded domains. By adapting the proof of the maximum principle for bounded C^2 functions, Theorem 5.3, and appealing to Theorem 8.8, instead of Theorem 5.1, we obtain

Theorem 8.15 (Weak maximum principle for bounded $H^1(\mathcal{O}, \mathfrak{w})$ functions on unbounded domains). *Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a possibly unbounded domain such that the divergence theorem holds. Assume the hypotheses of Theorem 8.8 (power weights and subdomains of \mathbb{H}) or Theorem 8.11 (general weights and subdomains of \mathbb{R}^d). Assume, in addition, that \tilde{c} obeys¹⁶ (2.14) a.e. on \mathcal{O} and that $a^{ij}, d^j, \mathfrak{w}, \vartheta$ obey (8.26), (8.27), (8.28), (8.29), (8.30), and*

$$\int_{\mathcal{O}} (1 + |x|^2)(1 + \vartheta^2) \mathfrak{w} \, dx < \infty, \quad (8.36)$$

and that (5.7) is obeyed a.e. on \mathcal{O} by (a, \tilde{b}) in place of (a, b) , where \tilde{b} is given by (8.24). Suppose $f \in L^2(\mathcal{O}, \mathfrak{w})$ and $\sup_{\mathcal{O}} f < \infty$. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ obeys (6.6) and (6.7) with $g = 0$ (when $\partial\mathcal{O} \setminus \Sigma$ non-empty) and

$$\operatorname{ess\,sup}_{\mathcal{O}} u < \infty, \quad (8.37)$$

then

$$\operatorname{ess\,sup}_{\mathcal{O}} u \leq 0 \vee \frac{1}{c_0} \operatorname{ess\,sup}_{\mathcal{O}} f.$$

Moreover, \mathfrak{a} has the weak maximum principle property on $\mathcal{O} \cup \Sigma$ in the sense of Definition 6.3.

Proof. We proceed almost exactly as in the proof of Theorem 5.3 and choose

$$M := 0 \vee \operatorname{ess\,sup}_{\mathcal{O}} (f + 2Ku),$$

where $K > 0$ is the constant arising in the proof of Theorem 5.3. Our hypotheses on f and u imply that $0 \leq M < \infty$. For a constant $\lambda \geq 0$, set

$$\mathfrak{a}_{\lambda}(u_1, u_2) := \mathfrak{a}(u_1, u_2) + \lambda(u_1, u_2)_{L^2(\mathcal{O}, \mathfrak{w})} \quad \forall u_1, u_2 \in H^1(\mathcal{O}, \mathfrak{w}).$$

Let v_0 be as in (5.8) and note that $v_0 \in H^2(\mathcal{O}, \mathfrak{w})$ by (8.35) and (8.36). For $\delta > 0$, choose $w \in H^1(\mathcal{O}, \mathfrak{w})$ as in (5.10) and observe that for all $v \in H_0^1(\mathcal{O} \cup \Sigma, \mathfrak{w})$ with $v \geq 0$ a.e. on \mathcal{O} ,

$$\begin{aligned} \mathfrak{a}_{2K}(w, v) &= \mathfrak{a}_{2K}(u, v) - \delta \mathfrak{a}_{2K}(v_0, v) - (c_0 + 2K)^{-1} \mathfrak{a}_{2K}(M, v) \quad (\text{by (5.10)}) \\ &= \mathfrak{a}(u, v) + 2K(u, v)_{L^2(\mathcal{O}, \mathfrak{w})} - \delta((A + 2K)v_0, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &\quad - (c_0 + 2K)^{-1}((A + 2K)M, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &\quad (\text{since } v_0, M \in H^2(\mathcal{O}, \mathfrak{w}) \text{ and applying (8.31)}) \\ &= \mathfrak{a}(u, v) + 2K(u, v)_{L^2(\mathcal{O}, \mathfrak{w})} - \delta((A + 2K)v_0, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &\quad - (c_0 + 2K)^{-1}((c + 2K)M, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &\leq (f, v)_{L^2(\mathcal{O}, \mathfrak{w})} + 2K(u, v)_{L^2(\mathcal{O}, \mathfrak{w})} - (M, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &\quad (\text{by (2.14), (5.9), and (6.6)}) \\ &\leq \left(\operatorname{ess\,sup}_{\mathcal{O}} (f + 2Ku)^+, v \right)_{L^2(\mathcal{O}, \mathfrak{w})} - (M, v)_{L^2(\mathcal{O}, \mathfrak{w})} = 0, \end{aligned}$$

where the final equality follows from the definition of M . We now apply Theorem 8.8 or 8.11 (instead of Theorem 5.1) and the remainder of the proof is the same as that of Theorem 5.3. \square

¹⁶This is equivalent to (6.9).

APPENDIX A. WEAK MAXIMUM PRINCIPLE FOR UNBOUNDED FUNCTIONS

Theorems 2.20 and 6.16 and Corollary 6.17 gave sufficient conditions describing when the weak maximum principle holds to unbounded $C^2(\mathcal{O})$, $H^1(\mathcal{O}, \mathfrak{w})$, or $H^2(\mathcal{O}, \mathfrak{w})$ functions, respectively, obeying a growth condition (2.17) defined by a function φ on the unbounded domain, $\mathcal{O} \subseteq \mathbb{R}^d$. In this section, we give examples of such growth conditions in Theorem 2.20, primarily in the case of the Heston operator, A .

We first compute the coefficients of the operator B in (2.15) explicitly, when A is as in (1.3). For $v \in C^2(\mathcal{O})$,

$$\begin{aligned} [A, \varphi]v &= A(\varphi v) - \varphi Av \\ &= (-a^{ij}\varphi_{x_i x_j} - b^i\varphi_{x_i} + c\varphi)v + (-a^{ij}(\varphi_{x_i}v_{x_j} + \varphi_{x_j}v_{x_i} + \varphi v_{x_i x_j}) - b^i\varphi v_{x_i}) - \varphi Av \\ &= -a^{ij}(\varphi_{x_i}v_{x_j} + \varphi_{x_j}v_{x_i}) - (a^{ij}\varphi_{x_i x_j} + b^i\varphi_{x_i})v \\ &= -(a^{ij} + a^{ji})\varphi_{x_j}v_{x_i} - (a^{ij}\varphi_{x_i x_j} + b^i\varphi_{x_i})v \\ &= -(a^{ij} + a^{ji})\varphi^{-1}\varphi_{x_j}((\varphi v)_{x_i} - \varphi_{x_i}v) - (a^{ij}\varphi_{x_i x_j} + b^i\varphi_{x_i})v \\ &= -(a^{ij} + a^{ji})(\log \varphi)_{x_j}(\varphi v)_{x_i} - (a^{ij}\varphi_{x_i x_j} + b^i\varphi_{x_i} - (a^{ij} + a^{ji})(\log \varphi)_{x_j}\varphi_{x_i})v, \end{aligned}$$

and therefore, since $B(\varphi v) = -[A, \varphi]v$ by (2.15), we see that for all $v \in C^2(\mathcal{O})$,

$$\begin{aligned} Bv &\equiv f^i v_{x_i} + f^0 v \equiv (a^{ij} + a^{ji})(\log \varphi)_{x_j} v_{x_i} \\ &\quad + (a^{ij}\varphi^{-1}\varphi_{x_i x_j} + b^i(\log \varphi)_{x_i} - (a^{ij} + a^{ji})(\log \varphi)_{x_j}(\log \varphi)_{x_i})v. \end{aligned} \tag{A.1}$$

Next, we give some examples of choices of functions, φ .

Example A.1 (Exponential-affine growth). When φ has the form

$$\varphi(x) = e^{-\langle h, x \rangle}, \quad \forall x \in \mathbb{R}^d, \tag{A.2}$$

for a fixed vector, $h \in \mathbb{R}^d$, and positive constant, C , the expression (A.1) for Bv simplifies to

$$Bv = -(a^{ij} + a^{ji})h_j v_{x_i} + (a^{ij}h_i h_j - b^i h_i - (a^{ij} + a^{ji})h_i h_j)v,$$

and thus,

$$Bv = -(a^{ij} + a^{ji})h_j v_{x_i} - (b^i h_i + a^{ij}h_i h_j)v, \quad v \in C^2(\mathcal{O}). \tag{A.3}$$

Therefore, $\hat{A} = (A + B)$ is given by

$$\hat{A}v = -a^{ij}v_{x_i x_j} - (b^i + (a^{ij} + a^{ji})h_j)v_{x_i} + (c - b^i h_i - a^{ij}h_i h_j)v. \tag{A.4}$$

Hence, when φ is as in (A.2), it is easy to tell when \hat{A} obeys the hypotheses of Theorem 2.20. Indeed, it suffices to ensure that the coefficient,

$$\hat{c} := c - b^i h_i - a^{ij}h_i h_j,$$

in the expression for \hat{A} obeys (2.14) for suitable h . \square

Example A.2 (Elliptic Heston operator and exponential-affine growth). Suppose $h = (h_1, h_2) = (L, N)$, where $L \geq 0$ and $N \geq 0$. From the identification of the coefficients, (a, b, c) , for the elliptic Heston operator, A , in Remark 5.4, we see that

$$\begin{aligned} \hat{c} &= c - b^i h_i - a^{ij}h_i h_j \\ &= r - (r - q - y/2)L - \kappa(\theta - y)N - \frac{y}{2}(L^2 + 2\rho\sigma LN + \sigma^2 N^2) \\ &= \frac{y}{2}(L + 2\kappa N - L^2 - 2\rho\sigma LN - \sigma^2 N^2) + r - \kappa\theta N - (r - q)L \end{aligned}$$

Therefore, provided the coefficients obey

$$L + 2\kappa N - L^2 - 2\rho\sigma LN - \sigma^2 N^2 \geq 0 \quad \text{and} \quad r - \kappa\theta N - (r - q)L > 0, \quad (\text{A.5})$$

we see that \hat{c} obeys (2.14), as desired. \square

Example A.3 (Exponential-quadratic growth). When φ has the form

$$\varphi(x) = e^{-L|x|^2}, \quad \forall x \in \mathbb{R}^d, \quad (\text{A.6})$$

for some positive constant, L , the expression (A.1) for Bv simplifies to give

$$Bv = -2L(a^{ij} + a^{ji})x_j v_{x_i} + (2L^2 a^{ij} \delta_{ij} - 2Lb^i x_i - 4L^2(a^{ij} + a^{ji})x_i x_j)v.$$

Therefore, in this case, \hat{A} is given by

$$\begin{aligned} \hat{A}v = & -a^{ij}v_{x_i x_j} - (b^i + 2L(a^{ij} + a^{ji})x_j)v_{x_i} \\ & + (c + (2L^2 a^{ij} \delta_{ij} - 2Lb^i x_i - 4L^2(a^{ij} + a^{ji})x_i x_j))v. \end{aligned} \quad (\text{A.7})$$

When φ is as in (A.6), one can see that \hat{A} will *not* obey the hypotheses of Theorem 2.20, in particular the condition (2.14), unless $L = 0$. \square

APPENDIX B. A WEIGHTED SOBOLEV INEQUALITY DUE TO V. MAZ'YA

The following Sobolev inequality and corollary provide an alternative insight into Corollary 8.6 and why it should not be expected to hold if the weights $x_d^{\beta-1}$ and $x_d^{\beta-1+\alpha}$ are replaced by more general weights. Let A_p , $1 \leq p < \infty$, denote the class of *Muckenhoupt weights* in the sense of [71, Definition 1.2.1 & 1.2.2].

Theorem B.1. [71, Theorem 2.6.1] *Let $w \in A_1$ and let $1 \leq q < \infty$. Suppose that μ is a positive Radon measure, satisfying*

$$M := \sup_{a \in \mathbb{R}^d, r > 0} \frac{r\mu(B_r(a))^{1/q}}{w(B_r(a))} < \infty. \quad (\text{B.1})$$

Then there is a constant $C = C'M$, where C' only depends on d, q and the A_1 constant for w , such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^d)$,

$$\|u\|_{L^q(\mathbb{R}^d, \mu)} \leq C \|Du\|_{L^1(\mathbb{R}^d, w)}. \quad (\text{B.2})$$

Conversely, if there exists a constant C such that (B.2) holds for every $u \in C_0^\infty(\mathbb{R}^d)$, then $C \geq C'M$, with C' as before and, in particular, M is finite.

Corollary B.2. *Assume the hypotheses of Theorem B.1 hold with $1 \leq q < \infty$. If $q > 1$, $1 \leq p < q/(q-1)$, and $q \leq s < \infty$ is defined by $s = pq/(p - q(p-1))$, then for all $u \in C_0^\infty(\mathbb{R}^d)$,*

$$\|u\|_{L^s(\mathbb{R}^d, \mu)} \leq C_1 \|Du\|_{L^p(\mathbb{R}^d, w)}, \quad (\text{B.3})$$

where $C_1 = pC/(p - q(p-1))$ and C is as in Theorem B.1.

Proof. When $p = 1$, then (B.3) follows from (B.2) with $s = q$, so we may suppose $p > 1$. For $1 < p' < \infty$ defined by $1/p + 1/p' = 1$, so $p' = p/(p-1)$, and $\gamma > 1$, we have

$$\begin{aligned} \| |u|^\gamma \|_{L^q(\mathbb{R}^d, \mu)} & \leq \gamma C \| |u|^{\gamma-1} Du \|_{L^1(\mathbb{R}^d, w)} \\ & \leq \gamma C \| |u|^{\gamma-1} \|_{L^{p'}(\mathbb{R}^d, \mu)} \| Du \|_{L^p(\mathbb{R}^d, w)}. \end{aligned}$$

Now choose γ to obey $\gamma q = s = (\gamma-1)p'$, so that

$$\gamma = \frac{p}{p - q(p-1)} \quad \text{and} \quad s = \frac{pq}{p - q(p-1)}.$$

The preceding inequality becomes

$$\|u\|_{L^s(\mathbb{R}^d, \mu)}^\gamma \leq \gamma C \|u\|_{L^s(\mathbb{R}^d, \mu)}^{\gamma-1} \|Du\|_{L^p(\mathbb{R}^d, \mu)},$$

and this yields (B.3). \square

It is useful to check Theorem B.1 when $w = 1$ and $\mu = dx$. Then $\mu(B_r(a)) = w(B_r(a)) \sim r^d$, so M in (B.2) is finite if and only if $\sup_{r>0} r^{1+d/q-d} < \infty$, so $1 - d(1 - 1/q) = 0$ and thus $q = d/(d-1)$, to give the usual Sobolev inequality (compare [40, Theorem 7.10]) when $p = 1$ and hence when $1 \leq p < d$ and $d/(d-1) \leq s < \infty$ defined by $s = dp/(d-p)$.

Note that to have $s > 2$ in (B.3), we need $p > 2q/(3q-2)$. If we also impose a constraint $p \leq 2$, then we must have $q > 1$, which holds in the basic case $w = 1$ and $\mu = dx$.

APPENDIX C. WEAK MAXIMUM PRINCIPLES OF FICHERA, OLEĬNIK, AND RADKEVIČ AND THE ELLIPTIC HESTON OPERATOR

We compare the weak maximum principles and uniqueness theorems provided by our article with those of Fichera, OleĬnik, and Radkevič [63] in the case of the elliptic Heston operator, A , in Example 1.2 on a domain $\mathcal{O} \subseteq \mathbb{H}$ and show that those of Fichera, OleĬnik, and Radkevič are strictly weaker when $0 < \beta < 1$.

C.1. Verification that the Heston operator and bilinear map coefficients have the required properties. We shall first illustrate how to choose \mathfrak{w} so that (8.2) holds for the elliptic Heston operator, A , on $\mathcal{O} \subseteq \mathbb{H}$. Denoting $(x, y) = (x_1, x_2)$, we have

$$\begin{aligned} Au &= -\frac{y}{2} (u_{xx} + 2\rho\sigma u_{xy} + \sigma^2 u_{yy}) - (r - q - y/2)u_x - \kappa(\theta - y)u_y + ru \\ &= -\frac{1}{2} \left((yu_x + y\rho\sigma u_y)_x + (y\rho\sigma u_x + y\sigma^2 u_y)_y \right) \\ &\quad + \frac{1}{2} (\rho\sigma u_x + \sigma^2 u_y) - (r - q - y/2)u_x - \kappa(\theta - y)u_y + ru \\ &= -\frac{1}{2} \left((yu_x + y\rho\sigma u_y)_x + (y\rho\sigma u_x + y\sigma^2 u_y)_y \right) \\ &\quad + (\rho\sigma/2 - (r - q - y/2))u_x + (\sigma^2/2 - \kappa(\theta - y))u_y + ru \\ &= -a^{ij}u_{x_i x_j} - \tilde{b}^i u_{x_i} + cu. \end{aligned}$$

Hence,

$$a = \frac{y}{2} \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} -\rho\sigma/2 + (r - q - y/2) \\ -\sigma^2/2 + \kappa(\theta - y) \end{pmatrix}, \quad c = r.$$

As in [18], we choose

$$\vartheta = \frac{y}{2}(1 - |\rho|) \min\{1, \sigma^2\} \quad \text{and} \quad \mathfrak{w} = y^{\beta-1} e^{-\gamma|x| - \mu y},$$

so that $\log \mathfrak{w} = (\beta - 1) \log y - \gamma|x| - \mu y$ and

$$(\log \mathfrak{w})_x = -\gamma \operatorname{sign}(x) \quad \text{and} \quad (\log \mathfrak{w})_y = (\beta - 1)y^{-1} - \mu.$$

The coefficient matrix, a , weight \mathfrak{w} , and degeneracy coefficient, ϑ , clearly obey the regularity conditions (8.26), (8.28), (8.29), and (8.30).

Recalling that $\beta = 2\kappa\theta/\sigma^2$ and $\mu = 2\kappa/\sigma^2$ and recalling the definition (8.24) of the coefficients \tilde{b}^i , we see that

$$\begin{aligned} b^1 &= \tilde{b}^1 - \vartheta(\log \mathfrak{w})_x a^{11} - \vartheta(\log \mathfrak{w})_y a^{12} \\ &= -\varrho\sigma/2 + (r - q - y/2) + \frac{\gamma}{2}y \operatorname{sign}(x) - \frac{1}{2}(\beta - 1 - \mu y)\varrho\sigma \\ &= (r - q - y/2) + \frac{\gamma}{2}y \operatorname{sign}(x) - \frac{\varrho\kappa}{\sigma}(\theta - y) \\ &= \left(r - q - \frac{\varrho\kappa\theta}{\sigma}\right) + y \left(\frac{\gamma}{2} \operatorname{sign}(x) + \frac{\varrho\kappa}{\sigma} - \frac{1}{2}\right), \end{aligned}$$

while

$$\begin{aligned} b^2 &= \tilde{b}^2 - \vartheta(\log \mathfrak{w})_x a^{21} - \vartheta(\log \mathfrak{w})_y a^{22} \\ &= -\sigma^2/2 + \kappa(\theta - y) + \frac{\gamma}{2}y \operatorname{sign}(x)\varrho\sigma - \frac{1}{2}(\beta - 1 - \mu y)\sigma^2 \\ &= \frac{\gamma}{2}y \operatorname{sign}(x)\varrho\sigma. \end{aligned}$$

The resulting bilinear map agrees with that in [18, Definition 2.2]. By making use of an affine change of variables [18, Lemma 2.2], we may assume that $r - q - \varrho\kappa\theta/\sigma = 0$ and so the expression for b^1 simplifies to

$$b^1 = y \left(\frac{\gamma}{2} \operatorname{sign}(x) + \frac{\varrho\kappa}{\sigma} - \frac{1}{2} \right).$$

We can easily see that the coefficients, (a, b, c, d) , of the bilinear map, \mathfrak{a} , associated with the elliptic Heston operator now obey the conditions (7.13), (8.2), (8.3), and (7.12); note that $(d^j) = 0$.

C.2. Comparison with the weak maximum principles and uniqueness theorems of Fichera. In the framework of Fichera (see [63, p. 308]), we let¹⁷ Σ denote the subset of points $x \in \partial\mathcal{O}$ where $a^{ij}(x)n_in_j = 0$ (with \vec{n} denoting the *inward*-pointing unit normal vector field, as in [63, p. 308]) and the *Fichera function* [63, Equations (1.1.2) & (1.1.3)] (taking into account our sign convention in (1.3) for the coefficients (a, \tilde{b}, c) of A) is

$$b_0 := \left(\tilde{b}^k - a_{x_j}^{kj}\right) n_k = \left(b^k + (\log \mathfrak{w})_{x_j} a^{kj}\right) n_k.$$

Following [63, p. 308], we denote by $\Sigma_1 \subset \Sigma$ the subset where $b_0 > 0$, by $\Sigma_2 \subset \Sigma$ the subset where $b_0 < 0$, and by $\Sigma_0 \subset \Sigma$ the subset where $b_0 = 0$; the set $\partial\mathcal{O} \setminus \Sigma$ is denoted by Σ_3 . By [63, Theorem 1.1.1], the characterization of the subsets $\Sigma, \Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3$ of the boundary $\partial\mathcal{O}$ remains invariant under smooth changes of the independent coordinates, (x_1, \dots, x_d) .

In our example, we have $\Sigma = \partial\mathcal{O} \cap \partial\mathbb{H}$ and $\vec{n} = (0, 1)$ along Σ , so that¹⁸

$$\begin{aligned} b_0(x, 0) &= b^2(x, 0) + (\log \mathfrak{w})_{x_j} a^{2j} \equiv \tilde{b}^2(x, 0) \\ &= -\sigma^2/2 + \kappa\theta = \frac{\sigma^2}{2}(\beta - 1). \end{aligned}$$

¹⁷In the work of Fichera [35, 58, 63, 64], the boundary of the domain $\mathcal{O} \subset \mathbb{R}^d$ is usually denoted by Σ and Σ^0 is the subset of points $x \in \Sigma$ where $a^{ij}(x)n_in_j = 0$.

¹⁸From [63, p. 310], one has that $\partial\mathcal{O} \cap \partial\mathbb{H}$ is given by $y = 0$ and $-Ay = -\sigma^2/2 + \kappa(\theta - y) = b_0$, and thus $\Sigma'_i = \Sigma_i$ for $i = 0, 1, 2$ in the notation of [63, p. 310].

Hence,

$$\partial\mathcal{O} \cap \partial\mathbb{H} = \begin{cases} \Sigma_2 & \text{if } 0 < \beta < 1, \\ \Sigma_1 & \text{if } \beta > 1, \\ \Sigma_0 & \text{if } \beta = 1, \end{cases}$$

while $\Sigma_3 = \mathbb{H} \cap \partial\mathcal{O}$. The *first boundary value problem of Fichera* [63, Equations (1.1.4) & (1.1.5)] for the operator A is to find a function $u \in C^2(\mathcal{O})$ such that

$$Au = f \text{ on } \mathcal{O}, \quad u = g \text{ on } \Sigma_2 \cup \Sigma_3,$$

given a source function f on \mathcal{O} and a boundary data function g on $\Sigma_2 \cup \Sigma_3$. But

$$\Sigma_2 \cup \Sigma_3 = \begin{cases} \partial\mathcal{O} & \text{if } 0 < \beta < 1, \\ \mathbb{H} \cap \partial\mathcal{O} & \text{if } \beta \geq 1. \end{cases}$$

Thus, for the Heston operator, the first boundary value problem of Fichera becomes

$$Au = f \text{ on } \mathcal{O}, \quad u = g \text{ on } \begin{cases} \partial\mathcal{O} & \text{if } 0 < \beta < 1, \\ \mathbb{H} \cap \partial\mathcal{O} & \text{if } \beta \geq 1. \end{cases}$$

Therefore, we see that the first boundary value problem of Fichera differs from the formulations in [18, 19, 20, 30] when $0 < \beta < 1$, where a Dirichlet boundary condition along $\partial\mathcal{O} \cap \partial\mathbb{H}$ is replaced by the requirement that u have a regularity property, in a weighted Hölder or Sobolev sense, up to the boundary portion $\partial\mathcal{O} \cap \partial\mathbb{H}$ which is strictly weaker than that of the Fichera maximum principles [63, Theorem 1.1.2 and Theorems 1.5.1 & 1.5.5] for $C^2(\mathcal{O})$ or $H_{\text{loc}}^1(\mathcal{O})$ functions, respectively. Note that

$$\Sigma_0 \cup \Sigma_1 = \begin{cases} \emptyset & \text{if } 0 < \beta < 1, \\ \partial\mathbb{H} \cap \partial\mathcal{O} & \text{if } \beta \geq 1. \end{cases}$$

In the case of $C^2(\mathcal{O})$ functions on bounded subdomains $\mathcal{O} \subset \mathbb{H}$, we note that the Fichera maximum principle for $C^2(\mathcal{O})$ functions [63, Theorem 1.1.2] requires that $u \in C^2(\mathcal{O} \cup \Sigma_0 \cup \Sigma_1) \cap C(\bar{\mathcal{O}})$ and $Au = f$ on $\mathcal{O} \cup \Sigma_0 \cup \Sigma_1$, which is *stronger* than the hypothesis of our Theorem 5.1 when $\beta \geq 1$, and yields, for $r > 0$,

$$\|u\|_{C(\bar{\mathcal{O}})} \leq \frac{1}{r} \|f\|_{C(\bar{\mathcal{O}})} \vee \|g\|_{C(\Sigma_2 \cup \Sigma_3)},$$

where¹⁹ $\Sigma_2 \cup \Sigma_3 = \mathbb{H} \cap \partial\mathcal{O}$ when $\beta \geq 1$ and $\Sigma_2 \cup \Sigma_3 = \partial\mathcal{O}$ when $0 < \beta < 1$. We see that the uniqueness result, when $f = 0$ on $\mathcal{O} \cup \Sigma_0 \cup \Sigma_1$, afforded by the Fichera maximum principle [63, Theorem 1.1.2] is *weaker* than that of our Theorem 5.1 when $0 < \beta < 1$, since we only require $g = 0$ on $\mathbb{H} \cap \partial\mathcal{O}$, and *not* $g = 0$ on $\partial\mathcal{O}$ to ensure that $u = 0$ on $\bar{\mathcal{O}}$. Indeed, the prescription of a Dirichlet boundary condition along $\partial\mathbb{H} \cap \partial\mathcal{O}$, when $0 < \beta < 1$, ensures that solutions to the first boundary value problem of Fichera are at most continuous up to $\partial\mathbb{H} \cap \partial\mathcal{O}$ and not smooth as in [18, 19, 20, 30].

Similar remarks apply to the Fichera maximum principle for weak solutions in $L^\infty(\mathcal{O})$ [63, Theorem 1.5.1 & 1.5.5]. Furthermore, our notions of weak solution, subsolution, or supersolution differ from those of [63, p. 318], which uses the adjoint operator A^* to define these concepts for

¹⁹There is a typographical error in the statement of [63, Theorem 1.1.2], where $\Sigma'_2 \cap \Sigma_3$ should be replaced by $\Sigma'_2 \cup \Sigma_3$; compare [58, Theorem 1.1.2]

functions $u \in L^\infty(\mathcal{O})$, with a bilinear map $(u, A^*v)_{L^2(\mathcal{O})}$ and space of test functions $v \in C^2(\bar{\mathcal{O}})$ with

$$v = 0 \text{ on } \begin{cases} \partial\mathcal{O} & \text{if } 0 < \beta < 1, \\ \mathbb{H} \cap \partial\mathcal{O} & \text{if } \beta \geq 1, \end{cases}$$

and thus implies a Dirichlet boundary condition along $\partial\mathbb{H} \cap \partial\mathcal{O}$, when $0 < \beta < 1$, which is redundant in our framework of weighted Hölder or Sobolev spaces.

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